Collective behavior in plasma

So far, we have considered the motion of a particle due to Lorentz force in the fields prescribed by a driver.

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

This "single particle motion" treatment will not be valid when many particles are considered because the impact of charge density and current density of these charges will become significant and can no longer be ignored.

Second, keeping track of individual particle motions and their effects on the fields around them is not practical. (Typical plasma contains $10^{10}-10^{23}$ particles; the fastest computers can follow $10^{10}-10^{12}$)

We will therefore have to come up with a new formalism to account for the behavior of many particles in electric and magnetic fields. In the next few lectures we develop the equations that govern the collective behaviors in plasma, which form the foundation of much of plasma physics.

We start from the fact that for practical purposes, we can consider particles point-like objects. The particle's properties then can be described in terms of delta functions:

$$\rho = q\,\delta(x-x')$$

Recall: δ function has the following important properties:

$$\delta(x-x') = \begin{cases} 0 & x\neq x' \\ \infty & x = x' \end{cases}$$

$$\int \delta(x-x')\,dx' = 1$$

$$\int \delta(x-x')\,f(x')\,dx' = f(x)$$

The total charge density for a collection of particles is given as the sum of all charge densities:

$$\rho = \sum_{j=1}^{N} q^j \, \delta\left(\vec{x} - \vec{x}^j(t)\right) \qquad \vec{x}^j(t): \text{ trajectory of the } j\text{th particle}$$

Total current density is then given by

$$\vec{J} = \sum_{j=1}^{N} q^j \, \vec{v}^j(t) \, \delta\left(\vec{x} - \vec{x}^j(t)\right)$$

<u>Phase Space</u>
A highly advantageous structure for the study of plasma is to take advantage of the concept of the phase space.

In real space, we consider the impact of forces on particles and then follow them in the space. In this case, position and velocity for a particle are both functions of time only:

$$\frac{d\vec{v}}{dt} \text{ is given by force } \vec{F} \xrightarrow{\text{solve for}} \begin{cases} \vec{x} = \vec{x}(t) \\ \vec{v} = \vec{v}(t) \end{cases}$$

In the phase space, we consider the impact of forces on particles of various velocities at a point in space. In this treatment, rather than following particles, we look at the variation of properties at a certain point. Therefore, velocity becomes an independent variable, as particles flowing in and out of a position in space may have a variety of velocities, and the forces on these particles will depend on this variable too:

<u>Real space</u>	<u>Phase Space</u>
$\vec{x}$ 3D space + time ← independent variables →	$(\vec{x}, \vec{v})$ 6 dimensions + t
n: density	F is phase space density
$\int d\vec{x}\, n = \#$ of particles	$\int d\vec{x}\, d\vec{v}\, F = \#$ of particles
$n = \sum_j \delta\left(\vec{x} - \vec{x}^j(t)\right)$	$F = \sum_j \delta\left(x - x^j(t)\right) \delta\left(\vec{v} - \vec{v}^j(t)\right)$

individual particles will change their position & velocity as a function of time

We derive an equation for the phase space density "*F*" using the continuity equation in the real space "*n*" as an aspiration. One can think of continuity equation as a conservation law for the number of particles:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \text{Flux of particles} = 0$$

e.g. $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Proof: $$\frac{\partial}{\partial t} \sum_j q^j \delta(\vec{x} - x^j) + \vec{\nabla} \cdot \sum_j q^j \vec{v}^j(t)\, \delta(\vec{x} - \vec{x}^j) = 0$$

$$= \sum_j q^j \left(-\frac{d\vec{x}^j}{dt}\right) \cdot \vec{\nabla} \delta(\vec{x} - \vec{x}^j) + \sum_j q^j \vec{v}^j(t) \cdot \vec{\nabla} \delta(\vec{x} - \vec{x}^j)$$

$$= \sum_j q^j \underbrace{\left[-\frac{d\vec{x}_j}{dt} + \vec{v}^j\right]}_{=0} \cdot \vec{\nabla} \delta(\vec{x} - x^j) = \boxed{0} \checkmark$$

By analogy, the density of phase space must also satisfy a continuity equation:

$$\frac{\partial F}{\partial t} + \vec{\nabla} \cdot \text{Flux of } F = 0$$

includes $\vec{\nabla}_x$ & $\vec{\nabla}_v$ as the two are independent variables for F

this can be written as

$$\frac{\partial}{\partial t} \sum_j \delta(\vec{x} - \vec{x}^j)\, \delta(\vec{v} - \vec{v}^j) + \vec{\nabla}_x \cdot \sum_j \vec{v}^j \delta(\vec{x} - \vec{x}^j)\, \delta(\vec{v} - \vec{v}_j)$$

$$+\vec{\nabla}_v \cdot \sum_j \vec{a}^j \, \delta(\vec{x}-\vec{x}^j)\, \delta(\vec{v}-\vec{v}^j) = 0 \quad \cdots (1)$$

$\vec{a}^j = \frac{d\vec{v}^j}{dt}$: acceleration experienced by each particle

proof:

$$\sum_j -\frac{d\vec{x}_j}{dt} \cdot \nabla_x \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) - \frac{d\vec{v}_j}{dt} \cdot \nabla_v \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$

$$+ \sum_j \left[\vec{v}^j \cdot \nabla_x \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) + \vec{a}^j \cdot \nabla_v \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) \right]$$

$$= \sum_j \left[-\frac{d\vec{x}_j}{dt} + \vec{v}^j \right] \cdot \nabla_x \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$

$$+ \left[-\frac{d\vec{v}^j}{dt} + \vec{a}^j \right] \cdot \nabla_v \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$

$$= \boxed{0} \checkmark$$

So far everything appears trivial and nothing more than definition. We are interested in an equation for "*F*" and we can get that by using the properties of delta function:

$$\vec{v}^j \delta(\vec{v}-\vec{v}^j) = \vec{v}\, \delta(\vec{v}-\vec{v}^j)$$

← δ function is zero for all $v \neq v^j$

Variable velocity, not dependent on j anymore

Similarly, $$a^j = \frac{q^j}{m^j}\left(\vec{E}(\vec{x}^j) + \vec{v}^j \times \vec{B}(\vec{x}^j)\right) = a^j(\vec{x}^j, \vec{v}^j)$$

so, $$\vec{a}^j \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) = \vec{a}(\vec{x},\vec{v})\,\delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$

This allows us to extract the acceleration and velocity functions from eqn 1:

$$① \Rightarrow \frac{\partial}{\partial t} \underbrace{\sum_j \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)}_{=F} + \vec{\nabla}_x \cdot \sum_j \vec{v}\,\delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}_j)$$

$$+ \vec{\nabla}_v \cdot \sum_j \vec{a}\,\delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) = 0$$

$$\therefore \frac{\partial F}{\partial t} + \vec{\nabla}_x \cdot (\vec{v}F) + \vec{\nabla}_v \cdot (\vec{a}F) = 0 \quad \cdots ②$$

flux of phase space density

F is a fluid element & fields/forces exist even where there are no particles (in this sense, F itself is analogous to a field)

$$② \Rightarrow \frac{\partial F}{\partial t} + \vec{v}\cdot\vec{\nabla}_x F + \vec{a}\cdot\vec{\nabla}_v F + F\left[\vec{\nabla}_x\cdot\vec{v} + \vec{\nabla}_v\cdot\vec{a}\right] = 0$$

$\vec{\nabla}_x \cdot \vec{v} = 0$ since x & v are independent variables

$\vec{\nabla}_v \cdot \vec{a} = 0$ for E&M forces (Homework problem)

So, the conservation of particles/continuity equation can be written as

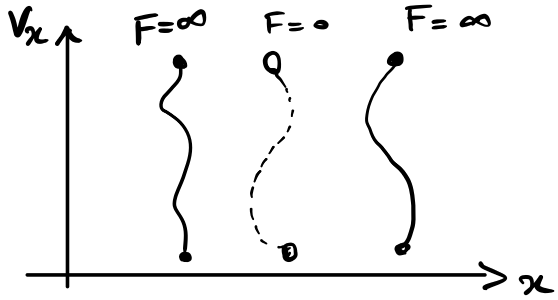
$$\boxed{\frac{\partial F}{\partial t} + \vec{v}\cdot\vec{\nabla}_x F + \vec{a}\cdot\vec{\nabla}_v F = 0}$$

This is called the Klimontovich equation. For the phase space then, the total time derivative following the trajectory of a particle is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_x + \vec{a}\cdot\vec{\nabla}_v$$

For our particular description so far, $\vec{F}$ is either zero or

∞ (at the location of a particle & does not change as phase space evolves)



$$\frac{DF}{Dt}=0$$

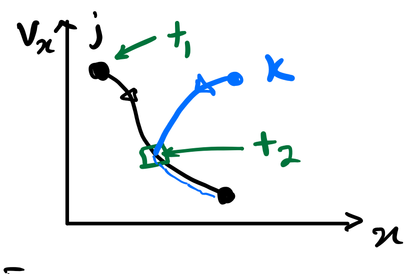
Note that the implication of the total derivative in time being zero is that there is no trajectory crossing in phase space. This is because

1. At the trajectory crossing, the two particles are at the same position, with the same velocity, experiencing the same force, which means that their trajectory would merge. But Newton's equations allow one to follow trajectory backwards as well as forwards in time, which means that running time backwards, it doesn't make physical sense for the trajectory of these two particles to "branch out" at t2. Therefore, the two particles will have the same trajectory for all time, meaning that trajectory crossing is not possible.
2. Mathematically is two trajectories cross, then the value of F at that point will increase to

$$F(\vec{x}^j, \vec{v}^j, t_2) = q^j \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j) + q^k \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$

Before crossing, the phase space density for jth particle was

$$F(\vec{x}^j, \vec{v}^j, t_1) = q^j \delta(\vec{x}-\vec{x}^j)\,\delta(\vec{v}-\vec{v}^j)$$



Therefore trajectory crossing would violate $\frac{DF}{Dt}=0$

This implies that the "fluid" is incompressible for point-like particles.
At this point, although we have introduced the new concept of phase space, conceptually everything is still the same, except that the forms are different:

Before: Maxwell's equations $(\vec{E}, \vec{B}, \rho, \vec{J})$

$$\rho = \sum_j q^j \delta(\vec{x} - \vec{x}^j)$$

$$\vec{J} = \sum_j q^j v^j \delta(\vec{x} - \vec{x}^j)$$

$$m\frac{d^2 x^j}{dt} = q^j(\vec{E} + \vec{v}^j \times \vec{B}) \quad \text{for every particle}$$

Now: Maxwell's equations $(\vec{E}, \vec{B}, \rho, \vec{J})$

$$F = \sum \delta(\vec{x} - \vec{x}^j)\, \delta(\vec{v} - \vec{v}_j)$$

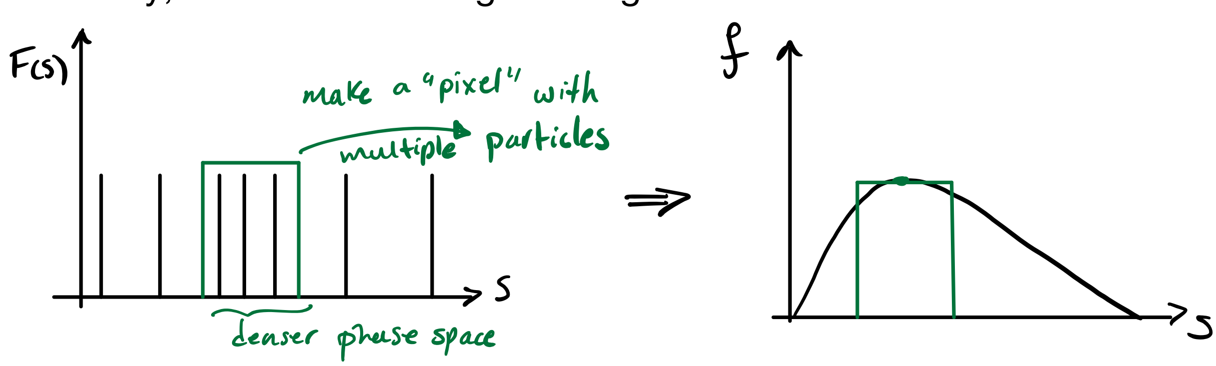
$$\frac{DF}{Dt} = 0 \quad \text{Klimontovich equation}$$

$$\rho(\vec{x}) = \int d\vec{v}\, q(\vec{x})\, F(\vec{x}, \vec{v})$$

$$\vec{J}(x) = \int d\vec{v}\, q(\vec{x})\, \vec{v}\, \vec{F}(\vec{x}, \vec{v})$$

In either case, calculating the trajectory of each particle is still required and this is not practical. To get a practical equation, we make the approximation where the exact function "*F*" (with all the delta functions) is replaced with a smooth function "*f*". There are several ways to view this transition, including the statical mechanical process of "ensemble averaging". This process is detailed in Warren's notes on pages 72(a)-73(b). I have included those pages in the appendix to these notes.

My preferred way of viewing it is that "*f*" is a "zoomed out" version of "*F*". So a "single point" or "pixel" of function "*f*" may include several particles. Practically, this is like creating a histogram of "*F*" for reasonable intervals:



i.e. $f(s)ds$ represents the number of particles between s &

s+ds, where ds is large enough to include many particles, but small enough to preserve the details of variation in density of particles along s.

If this process seems confusing, it is very instructive to reflect on how one would create a histogram for a quantity like grades or heights of people in a group and why those are useful concepts.

Now, the question is whether a version of Klimontovich equation also applies to smooth function, “*f*”? Physically, the phase density “*f*” is impacted by all the same processes that effect the change in “*F*” (i.e. flux of particles from elsewhere in position and velocity space), except that now many particles can inhabit the same “point” in the phase space for “*f*”. Therefore, there is an additional process that changes the phase space density “*f*”, and that comes from the collision of particles that inhabit the same point:

$$\left.\begin{aligned} \frac{Df}{Dt} &= \frac{D}{Dt}F + \left.\frac{\partial f}{\partial t}\right|_{collisions} \\ \frac{DF}{Dt} &= 0 \end{aligned}\right\} \Rightarrow \frac{Df}{Dt} = \underbrace{\left.\frac{\partial f}{\partial t}\right|_{collisions}}_{\propto \frac{1}{N_D}}$$

If $N_D \gg 1$, i.e. plasma is collisionless

$$\boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f + \vec{a}\cdot\vec{\nabla}_v f \cong 0}$$

This is the Vlasov equation, probably the most important equation in all of plasma physics.

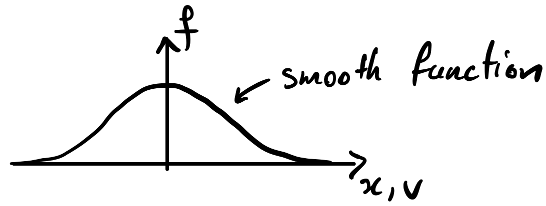
So, the Vlasov fluid, “*f*” is also an incompressible fluid. This equation is also called the collisionless Boltzmann equation.

Now, we have the Maxwell's equations ($\vec{E}, \vec{B}, \rho, \vec{J}$)

approximation valid for $N_D >> 1$

$$\begin{cases} \dfrac{Df}{Dt} = 0 \\ \rho = \int (d\vec{v})\, q\, f \\ \vec{J} = \int (d\vec{v})\, q\, \vec{v}\, f \end{cases}$$

$f(x,v,t) = f_0(x,v)$ at $t=0$ for each species



<u>Fluid Equations</u>
Sometimes the details of the distribution of velocities is not known. In this case, we can still learn about the behavior of the plasma by looking at the spatial fluid elements, which are derived by taking the <u>moments</u> of the Vlasov equation.

Recall that if you have a distribution function then

$$\int_{-\infty}^{\infty} d\vec{v}\, f(\vec{x}, \vec{v}, t) = n(\vec{x}, t) \text{ : density of particles}$$

Then, the average of any function *g(v)* is given by

$$\langle g(\vec{v}) \rangle = \frac{\int_{-\infty}^{\infty} d\vec{v}\, g(\vec{v})\, f(\vec{x},\vec{v},t)}{\int_{-\infty}^{\infty} d\vec{v}\, f(\vec{x},\vec{v},t)}$$

$$\Rightarrow \int_{-\infty}^{\infty} d\vec{v}\, g(\vec{v})\, f(\vec{x},\vec{v},t) = \langle g \rangle(\vec{x},t)\; n(\vec{x},t)$$

This integral is called moment of f if $g(\vec{v}) = \vec{v}^{\,m}$, where "m" is some integer

0^{th} moment of $\dfrac{Df}{Dt} = 0$

$$\int d\vec{v}\; \vec{v}^{\,0}\, \frac{Df}{Dt} = 0$$

$$\int d\vec{v}\, v_i \left[\frac{\partial f}{\partial t} + v_j \frac{\partial}{\partial x_j} f + a_j \frac{\partial}{\partial v_j} f \right] = 0$$

The subscript indicate the sum over repeated indices. This notation is known as Einstein notation and is a very powerful tool for simplifying vector calculus. A slight diversion is required here to explain this notation: Consider an orthogonal coordinate system with the unit vectors

$\hat{x}_1, \hat{x}_2, \hat{x}_3$ defined in this order:

$$\begin{cases} \hat{x}_1 \times \hat{x}_2 = \hat{x}_3 \\ \hat{x}_3 \times \hat{x}_1 = \hat{x}_2 \\ \hat{x}_2 \times \hat{x}_3 = \hat{x}_1 \end{cases}$$

$1 \to 2 \to 3 \to 1$ is defined as even permutations (+ve cross product)

$1 \to 3 \to 2 \to 1$ is defined as odd permutations (-ve cross product)

Any vector $\vec{A}$ can be written as

$$\vec{A} = A_1 \hat{x}_1 + A_2 \hat{x}_2 + A_3 \hat{x}_3$$

where A_i $(i = 1, 2, 3)$ are the components of the vector.

inner product: $\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_i A_i B_i$

Einstien Notation: $\vec{A} \cdot \vec{B} = \underbrace{A_i B_i}_{\text{Any repeated indices are to be summed over}}$

For cross products, we define the Levi-Civita tensor or so called permutation tensor, ϵ_{ijk}. This tensor is defined such that

$$\left[\vec{A} \times \vec{B}\right]_i = \sum_{j,k} \epsilon_{ijk} A_j B_k = \underbrace{\epsilon_{ijk} A_j B_k}_{\text{using Einstien notation}}$$

$\epsilon_{ijk} = 0$ if $i=j$, $j=k$, or $i=k$ (i.e. any repeated index)

$= 1$ for even permutations

$= -1$ for odd permutations

As it turns out, all vector geometry can be analyzed quite simply using this tensor and the identity

$$\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$$

where δ_{ij} is the Kroneker delta function:

$$\begin{cases} \delta_{ij} = 1 & \text{for } i=j \\ \delta_{ij} = 0 & \text{for } i \neq j \end{cases}$$

Example: $\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_i = \epsilon_{ijk}\, A_j\, [\vec{B} \times \vec{C}]_k$

$$= \underbrace{\epsilon_{ijk}}_{\text{even permutation}}\, A_j\, \epsilon_{klm}\, B_l\, C_m$$

$$= \epsilon_{kij}\,\epsilon_{klm}\, A_j\, B_l\, C_m$$

$$= \left(\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\right) A_j\, B_l\, C_m$$

$$= B_i\, A_m C_m - C_i\, A_j B_j$$

$$= \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B})$$

(known as BAC CAB rule!

We can also use this for vector calculus as well by considering the "del operator" as a vector: $\partial_i = \dfrac{\partial}{\partial x_i}$

So, $\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\,\partial_j\,(\vec{\nabla}\times\vec{A})_k = \epsilon_{ijk}\,\partial_j\,\epsilon_{klm}\,\partial_l A_m$

$$= \epsilon_{kij}\,\epsilon_{klm}\,\partial_j\partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l A_m$$

$$= \partial_i\,\partial_j A_j - \partial_j\partial_j\,A_i$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}}$$

From Warren's notes: try $\vec{B}\times\vec{\nabla}\times\vec{A}$ & $\vec{A}\times\vec{\nabla}\times\vec{A}$ for fun!

Ok, back to our regularly scheduled program! The Vlasov equation can be written in terms of the Einstein notation:

$$\frac{\partial f}{\partial t} + v_j\frac{\partial}{\partial x_j}f + a_j\frac{\partial}{\partial v_j}f = 0$$

0th moment: $\displaystyle\int d\vec{v}\left\{\underset{①}{\frac{\partial f}{\partial t}} + \underset{②}{v_j\frac{\partial}{\partial x_j}f} + \underset{③}{a_j\frac{\partial}{\partial v_j}f}\right\} = 0$

term ①: $\displaystyle\int d\vec{v}\,\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d\vec{v}\,f = \frac{\partial}{\partial t}n$

term ②: $\displaystyle\int d\vec{v}\,v_j\frac{\partial}{\partial x_j}f = \frac{\partial}{\partial x_j}\underbrace{\int d\vec{v}\,v_j f} = \frac{\partial}{\partial x_j}\mathbf{V}_j\,n = \vec{\nabla}\cdot(n\vec{V})$

average velocity / fluid velocity

term ③: $\displaystyle\int d\vec{v}\,a_j\frac{\partial}{\partial v_j}f =$ integrate by parts $=$

$$= \int d\vec{v}\, \frac{\partial}{\partial v_j}(a_j f) - \underbrace{\int d\vec{v} f \frac{\partial}{\partial v_j} a_j}$$

$\downarrow$ Gauss's Law $\quad \vec{\nabla}_v \cdot \vec{a} = 0$ for E&M force (HW)

$$= \oint d\vec{s}_v \cdot \vec{a} f$$

first term can be considered a volume integral in "v space" & therefore converted to surface integral with boundary at ∞.

for any physical system, $f \to 0$ as $v \to \infty$, so the first term is also zero.

$$① + ② + ③ = 0$$

$$\therefore \boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{V}) = 0}$$ continuity equation/conservation of particles.

Note 1: This applies to each individual species, since each species could have a different distribution function

Note 2: $\vec{V}$ in this equation is an average velocity at each position. because of the dependence of "f" on (x,t), which is used in its derivation, $\vec{V}$ (the fluid velocity) is not an independent variable & depends on (x,t).

To solve this continuity equation, we need to know how fluid velocity evolves. To find that, we need it take the first moment of the Vlasov equation:

$$\int (d\vec{v})\, m v_i \left[\underset{①}{\frac{\partial f}{\partial t}} + \underset{②}{v_j \frac{\partial}{\partial x_j} f} + \underset{③}{a_j \frac{\partial}{\partial v_j} f} \right] = 0$$

Term ①: $\int d\vec{v}\, m v_i \frac{\partial}{\partial t} f = \frac{\partial}{\partial t} \int d\vec{v}\, m v_i f = \frac{\partial}{\partial t} mn\vec{V}$

Term ②: $\int d\vec{v}\, m v_i v_j \frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j} \int d\vec{v}\, m v_i v_j f$

We write v_i in terms of average fluid velocity:

$$v_i = \underbrace{V_i(x,t)}_{\text{note: this is constant in variable "v"}} + v_{ri} \quad \text{--- ④}$$

Since $\int d\vec{v}\, v_i f = n V_i(x,t)$, if we average ④, we get:

$$\int d\vec{v}\, v_i f = \int d\vec{v}\, V_i(x,t) f + \int d\vec{v}\, v_{ri} f$$

$$n V_i(x,t) = V_i(x,t) \int d\vec{v}\, f + \int d\vec{v}\, v_{ri} f$$

$$n V_i(x,t) = n V_i(x,t) + \int d\vec{v}\, v_{ri} f$$

$$\therefore \boxed{\int d\vec{v}\, v_{ri} f = 0}$$

Physically, this means that each velocity can be written as the average velocity plus a remainder, where logically the average of the remainder should be zero.

Term ② becomes: $\frac{\partial}{\partial x_j} \int d\vec{v}\, m v_i v_j f$

$$= \frac{\partial}{\partial x_j} \int (d\vec{v})\, m (V_i + v_{ri})(V_j + v_{rj}) f$$

$$= \frac{\partial}{\partial x_j} \int (d\vec{v})\, m \left[V_i V_j + v_{ri} V_j + v_{rj} V_i + v_{ri} v_{rj} \right] f$$

$$= \frac{\partial}{\partial x_j}\left[m\bar{V}_i\bar{V}_j \int d\vec{v}\, f + m\bar{V}_j \cancel{\int d\vec{v}\, v_{ri} f}^{\,0} + m\bar{V}_i \cancel{\int d\vec{v}\, v_{rj} f}^{\,0} + m\int d\vec{v}\, v_{ri} v_{rj} f \right]$$

$$= \frac{\partial}{\partial x_j}\left[m\bar{V}_i\bar{V}_j n + 0 + 0 + P_{ij} \right]$$

$P_{ij} \equiv \int d\vec{v}\, m\, v_{ri} v_{rj} f$ represent the pressure / flow of momentum

Term ③ $\int d\vec{v}\, m v_i a_j \frac{\partial}{\partial v_j} f$ integration by parts again

$$= \int d\vec{v}\, \frac{\partial}{\partial v_j}(m v_i a_j f) - \int d\vec{v}\, m a_j f \frac{\partial}{\partial v_j} v_i - \int d\vec{v}\, m v_i f \frac{\partial}{\partial v_j} a_j$$

using Gauss's Law / Divergence Thm again

$\frac{\partial}{\partial v_j} a_j = 0$ again, HW problem

$$= 0 - \int d\vec{v}\, m a_j f \delta_{ij} - 0$$

$$= -\int d\vec{v}\, m a_i f$$

$$= -\int d\vec{v}\, m f \frac{q}{m}\left[E_i + \epsilon_{ijk} v_j B_k\right]$$

$$= -q E_i \underbrace{\int d\vec{v}\, f}_{=n} - q\,\epsilon_{ijk} B_k \underbrace{\int d\vec{v}\, f\, v_j}_{= n\bar{V}_j} = -nq\left[\vec{E} + \vec{\bar{V}} \times \vec{B}\right]$$

①+②+③=0

$$\Rightarrow \frac{\partial}{\partial t} mnV_i + \frac{\partial}{\partial x_j} mnV_iV_j + \frac{\partial}{\partial x_j} P_{ij} - nq\left[E_i + \epsilon_{ijk} V_j B_k\right] = 0$$

This is the equation of conservation of momentum. The first two terms can be further simplified:

$$\frac{\partial}{\partial t} mnV_i + \frac{\partial}{\partial x_j} mnV_i V_j = mn \frac{\partial}{\partial t} V_i + mnV_j \frac{\partial}{\partial x_j} V_i + mV_i \underbrace{\left[\frac{\partial}{\partial t} n + \frac{\partial}{\partial x_j} nV_j\right]}_{=0 \text{ from continuity eqn}}$$

Therefore, the conservation of momentum can be written as

$$\boxed{mn\left[\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}\right] = qn\left[\vec{E} + \vec{v}\times\vec{B}\right] - \vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}}$$

called Euler's equation

$$mn\left[\frac{\partial V_i}{\partial t} + V_j \frac{\partial}{\partial x_j} V_i\right] = qn\left[E_i + \epsilon_{ijk} V_j B_k\right] - \frac{\partial}{\partial x_j} P_{ij}$$

$\overleftrightarrow{\mathbb{P}} = P_{ij}$ is called pressure tensor.

The pressure tensor is the new unknown. In order to solve this equation, we need to know the evolution of the pressure, which means we need to get the next moment of the Vlasov equation!

2nd moment: $\int d\vec{v}\, m v_i v_j \left[\frac{\partial}{\partial t} f + v_k \frac{\partial}{\partial k} f + a_k \frac{\partial}{\partial v_k} f\right] = 0$

(HW problem)

Of course, this will need to a new unknown, which will need the next moment and so on. In general, the system of moment equations has an infinite number of equations, with each new moment defining a new quantity, the evolution of which is described by the next higher moment.

This chain is usually broken at the first moment using an "equation of state", which is a model for variation of the pressure tensor in terms of the other variables in the momentum conservation equation: $(n, \vec{V}, \vec{E}, \vec{B})$
Such a model allows us to "close" this system of equations.
The most common models used assume either an absence of heat flux (the adiabatic condition) or a constant temperature (isothermal gas law):

Adiabatic: $PV^{N+2/N} = \text{constant} = Pn^{-(N+2/N)}$

Isothermal: $P = nKT \qquad \frac{P}{n} = \text{constant}$

$$P_{ij} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \quad \text{isotropic, a function of } n$$

Finally, this allows us to obtain an approximate closed set of equations for the collective behavior of plasma, which are called the <u>Maxwell-Fluid Equations</u>.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \rho = q^j n^j$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \vec{J} = q^j n^j \vec{V}^j$$

$$\frac{\partial n^j}{\partial t} + \vec{\nabla}\cdot(n^j \vec{V}^j) = 0$$

$$m^j n^j \left[\frac{\partial \vec{V}^j}{\partial t} + (\vec{V}^j\cdot\vec{\nabla})\vec{V}^j \right] = q^j \left[\vec{E} + \vec{V}^j \times \vec{B} \right] - \vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}^j$$

$\mathbb{P}_{ij} \equiv \int d\vec{v}\, m\, v_{ri} v_{rj} f$ is assumed to be isotropic,

$\overleftrightarrow{\underline{\mathbb{P}}} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$ & its dependence on $n, \vec{V}, \vec{E}$ or $\vec{B}$ is reffered to as the equation of state. "Default" equation of state is the ideal gas law: $p = nKT$

For almost the rest of the class, we will explore the consequence these equations. Before we dive deep, let's look at some of the properties of these equations:

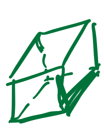
1. These are approximate, but they are based on sound physical rigor. In the book, they are introduced using plausibility arguments (to go from Klimontovich to Vlasov to Fluid equations.)
2. Each quantity is defined at a fixed location. In going from Vlasov to fluid equations, we average out the effect of velocity distribution. We are therefore looking at tiny volume elements and analyzing how the properties ascribed to such an element vary in space and evolve in time. This is called an Eulerian description.

n & $\vec{V}$ are average quantities in "velocity space" & are assigned to a fixed location. They are called the fluid density & fluid velocity.

continuity eqn: $\dfrac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{V}) = 0$

$$\Rightarrow \frac{\partial}{\partial t} \int d\vec{x}\, n = - \int d\vec{x}\, \vec{\nabla} \cdot (n\vec{V})$$

$$\Rightarrow \frac{\partial}{\partial t} \underbrace{N}_{\text{\# of particles in a fixed volume}} = - \oint \underbrace{d\vec{S} \cdot n\vec{V}}_{\text{particle flux into or out of the volume}}$$



\# of particles in a fixed volume

particle flux into or out of the volume

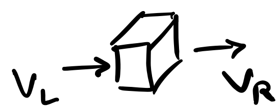
$$mn\left[\frac{\partial}{\partial t}\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = qn\left[\vec{E}+\vec{V}\times\vec{B}\right] - \vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}$$

$$\therefore\ \frac{\partial\vec{V}}{\partial t} = \underset{①}{-(\vec{V}\cdot\vec{\nabla})\vec{V}} + \underbrace{\frac{q}{m}\left[\vec{E}+\vec{V}\times\vec{B}\right]}_{\substack{\text{force/m}\\ ②}} - \underset{③}{\frac{1}{mn}\vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}}}$$

This equation says that the <u>average velocity</u> in a volume element (a “fluid pixel”) can change for these reasons:

① There is a flux of average velocity into out of it,

$-(\vec{V}\cdot\vec{\nabla})\vec{V}$



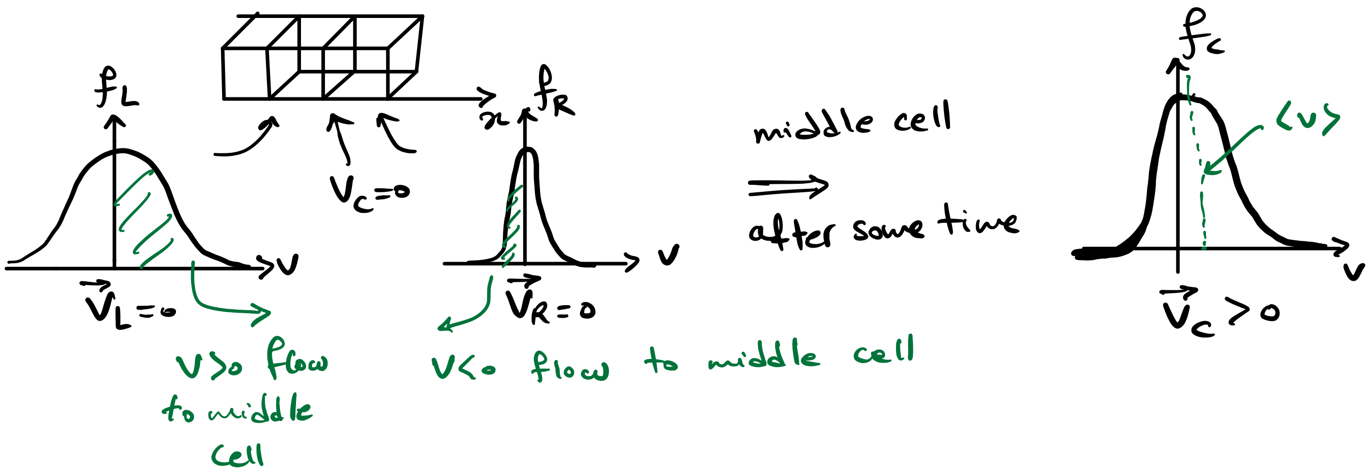
$\frac{\partial\vec{V}}{\partial t} > 0$ if $V_L > V_R$ for example.

② There is a force on all the particles in the fluid element. Note again that the fluid element is small enough that the force is assumed to be the same. If all velocities change, then the average velocity must change

③ There is a flux of random velocities

$\vec{V}(x,t)$ can change if $\vec{V_0}=0$ at $t=0$ everywhere.

for example consider 3 fluid element “pixels” with different initial distributions next to each other:



3. The pressure tensor is not necessarily isotopic. For example in a strong applied field (such as that of a laser), the pressure can be different in the direction parallel to the laser polarization and the direction perpendicular to it. i.e.

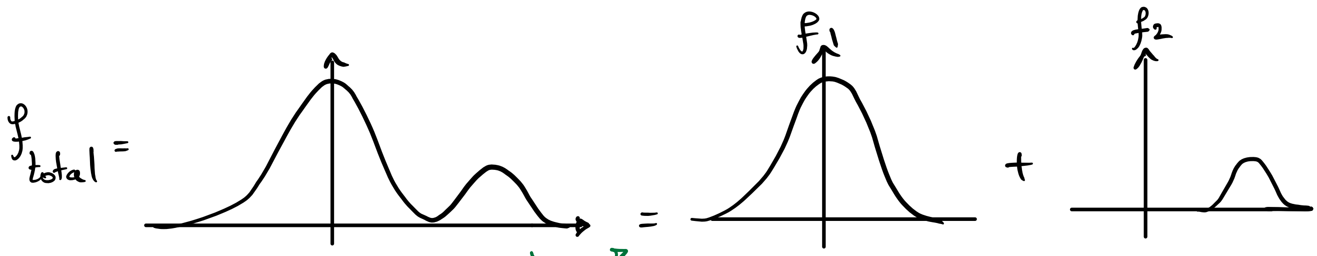
$$\underline{\mathbb{P}} = \begin{bmatrix} P_{\parallel} & 0 & 0 \\ 0 & P_{\perp} & 0 \\ 0 & 0 & P_{\perp} \end{bmatrix}$$

4. $\left\{ \frac{\partial}{\partial t} + \vec{V}(x,t) \cdot \vec{\nabla} \right\}$ is called the convective derivative

It represents the total derivative of velocity following the particles of a particular fluid element ("the pixels") in and out of that element (represented by $\frac{d}{dt}$ or $\frac{D}{Dt}$, total derivative)

There is a fluid description where these fluid elements are actually moving. Such a description is called Lagrangian description

5. A fluid equation exists for each "species" for which an "*f*" is specified. Therefore, we can always break up one species, for example electrons, into several species. For example,



we might want to break this up

Now, we will have fluid equations for both "species", *f1* and *f2*. This is a powerful technique that will allow us to study a number of instabilities.

Linearization of Fluid Equations

In the analysis of waves in plasma, we often make the simplifying assumption that the space and time vary as harmonics of fundamental spatial and temporal frequencies (see Appendix for the review of fundamental properties of waves). To do so, we need linear differential equations. Therefore, we will first linearize the Maxwell-Fluid equations. Implicit in this work is the assumption that the amplitude of the waves are "small".

Maxwell–Fluid Equations:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \sum_j q^j n^j \vec{V}^j + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\sum_j q^j n^j \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{V}) = 0 \\ &\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \frac{q}{m}(\vec{E} + \vec{V}\times\vec{B}) - \frac{\nabla p}{mn} \end{aligned}\right\} \text{for each species } j$$

The non-linear terms are

$$(\vec{V}\cdot\vec{\nabla})\vec{V} \qquad \vec{\nabla}\cdot(n\vec{V}) \qquad \frac{\nabla p}{mn}$$

Because the wave amplitude is going to be small, we write each parameter as an expansion around a dominant term. Physically, this means that the wave is a small modification on the “background” plasma:

$$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots$$

$$\vec{V} = \vec{V}_0 + \varepsilon \vec{V}_1 + \varepsilon^2 \vec{V}_2 + \cdots$$

$$\vec{E} = \vec{E}_0 + \varepsilon \vec{E}_1 + \varepsilon^2 \vec{E}_2 + \cdots$$

$\varepsilon \ll 1$ is called a smallness parameter & it is just to keep track of orders of magnitude. You could just as easily work only with n_0, n_1, n_2, etc, keeping in mind that $n_1 \ll n_0$ & so on.

For the background plasma, consider the case of

→ no applied electric field ($\vec{E}_0 = 0$)

→ possible magnetization ($\vec{B}_0 \neq 0$)

→ plasma approximation: $n_{0i} = n_{0e} = n_0$ with $\nabla n_0 = 0$

Euler's equation:

$$\frac{\partial}{\partial t} \sum_{i=0}^{\infty} \varepsilon^i \vec{V}_i + \left(\sum \varepsilon^i \vec{V}_i \cdot \vec{\nabla} \right) \sum \varepsilon^i \vec{V}_i = \frac{q}{m} \left[\sum \varepsilon^i \vec{E}_i + \sum \varepsilon^i \vec{V}_i \times \sum \varepsilon^i \vec{B}_i \right] - \frac{KT_e \nabla (\varepsilon^i n_i)}{m \sum \varepsilon^i n_i}$$

Now, gather and balance the terms for each order of ε:

$$\varepsilon^0: \frac{\partial}{\partial t}\vec{V}_0 + (\vec{V}_0 \cdot \vec{\nabla})\vec{V}_0 = \underset{(\vec{E}_0 = 0)}{0} + \frac{q}{m}\vec{V}_0 \times \vec{B}_0 + \underset{(\nabla n_0 = 0)}{0}$$

unlike the case of single particle motion, here we take the $V_0 = 0$ solution: physically, this means that the background plasma is at rest.

note: subscripts for each term add to <u>one</u>.

$$\varepsilon^1: \frac{\partial}{\partial t}\vec{V}_1 + \underset{}{\overset{(\vec{V}_0 = 0)}{0}} = \frac{q}{m}\left[\vec{E}_1 + \vec{V}_1 \times \vec{B}_0\right] - \frac{KT}{m n_0}\nabla n_1$$

$$\varepsilon^2: \frac{\partial}{\partial t}\vec{V}_2 + (\vec{V}_1 \cdot \vec{\nabla})\vec{V}_1 = \frac{q}{m}\left[\vec{E}_2 + \vec{V}_2 \times \vec{B}_0 + \vec{V}_1 \times \vec{B}_1\right] - \frac{KT}{m n_0}\nabla n_2$$

etc

Note that $\vec{V}_i, \vec{E}_i, \vec{B}_i$ are functions of $\vec{V}_{i-1}, \vec{E}_{i-1}, \vec{B}_{i-1}$ which were solved for in previous equations in an iterative process.

The complete set of <u>linear</u> Maxwell-Fluid equations is

$$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \qquad \vec{\nabla} \times \vec{B}_1 = \mu_0\left[e n_0 (\vec{V}_{1i} - \vec{V}_{1e})\right] + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B}_1 = 0 \qquad \vec{\nabla} \cdot \vec{E}_1 = \frac{e(n_{1i} - n_{1e})}{\epsilon_0}$$

$$\frac{\partial n_1}{\partial t} + n_0(\vec{\nabla} \cdot \vec{V}_1) = 0 \qquad \longleftarrow \text{for each species}$$

$$\frac{\partial}{\partial t}\vec{V}_1 = \frac{q}{m}\left(\vec{E}_1 + \vec{V}_1 \times \vec{B}_0\right) - \frac{\gamma KT}{m n_0}\nabla n_1$$ ← for each species

The equation of state determines γ: $\gamma = 1$ isothermal; $\gamma = \frac{2+N}{N}$ adiabatic

N: degrees of freedom

Note: in this system of equations, the fluid continuity equation is redundant and could be derived by taking the divergence of $\vec{\nabla}\times\vec{B}$ equation

We now have all the tools we need to start analyzing "small-amplitude" waves in plasma. We will look for the natural modes of the system by finding self-consistent solutions to the linearized Maxwell-Fluid equations.

Two approaches:

A) Fourier Analysis first: assume every wave quantity (those with subscript 1) have a form $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

e.g. $\frac{\partial}{\partial t}n_1 + n_0\vec{\nabla}\cdot\vec{V}_1 = 0 \Rightarrow -i\omega n_1 + n_0(i\vec{k}\cdot\vec{V}_1) = 0$

and then solve the resulting set of algebraic equations

B) Combine the original equations into a single equation for a wave quantity & then do Fourier analysis. This is preferred because it makes the wave-like solution explicit, but it is not always possible