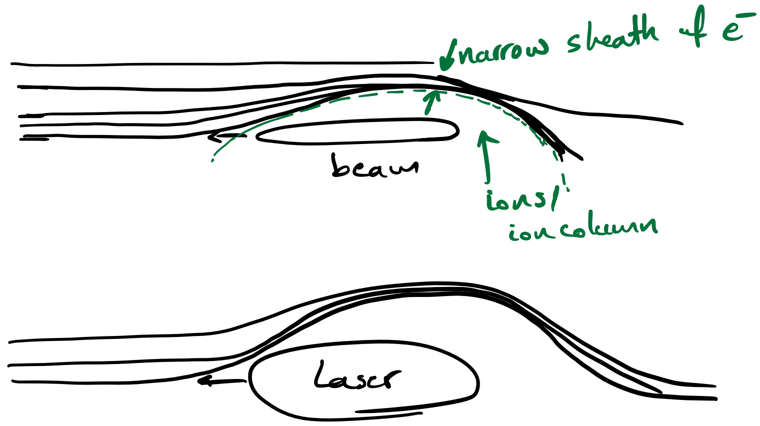
Problem Setup:

Suppose we have a particle beam or a laser. The laser ponderomotive force or the electron beam's Lorentz force pushes electrons forward, but also sideways. If you look at "streamlines" of electrons,



The electrons bunch up in a narrow sheath, creating an ion column. This is a very important regime because a majority of experiments that have been conducted have occurred in this regime.

Even though this structure is nonlinear, it can be very stable and has a number of properties that makes this structure very desirable for accelerating electrons. This "blowout" was first discovered by Rosenzweig, Breizman, Katsouleas, and Su (PRA, 44, R6189, 1991) at UCLA when looking at computer simulations of the interaction between an intense electron beam and plasma. The intention in this section is *not* to talk about how to excited, but about the properties of this structure, which is an azimuthally symmetric ion column surrounded by a narrow sheath around it. This was described by Wei Lu in 2006 in two papers(PRL 96, 165002 & PoP 13,056709), and expanded in 2021 (PPCF) with some interesting ideas and we are going to go through these in this class.

Setting up the problem: since most of the time we are interested on the forces of the wakefield on a trailing beam that is being accelerated, we can look at the forces of the wakefield on this trailing beam:

$\vec{V}_b \approx \hat{z}c$ i.e. Particle moves near the speed of Light along $\hat{z}$

$\therefore F_z = q E_z$ (since $v = \hat{z}$, so the effect of $\vec{B} = 0$)

$F_\perp = q(\vec{E}_\perp + (\hat{z} \times \vec{B})_\perp)$

$$\Rightarrow \begin{cases} F_x = q\,(E_x - cB_y) \\ F_y = q\,(E_y + cB_x) \end{cases}$$

We can rewrite these forces in very simple form if we use the relationship between $\vec{E}$ & $\vec{B}$ from Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\hat{x} : -\partial_y E_z - \partial_z E_y = \frac{\partial B_x}{\partial t}$$

using the relationships in the comoving frame, $\xi = ct - z$

$$\partial_z \to -\partial_\xi \qquad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} + c\frac{\partial}{\partial \xi} \to c\frac{\partial}{\partial \xi}$$ under the quasistatic approximation $(\partial_{t'} \to 0)$

$$\Rightarrow \hat{x} : \{\cancel{+}\, \partial_y E_z = \cancel{+}\, \partial_\xi (E_y + cB_x)\} \times q$$

$$\boxed{\partial_y F_z = \partial_\xi F_y} \;\ldots\; ①$$

By analogy, $\{\partial_x E_z = -\partial_\xi\, (E_x - cB_y)\} \times q$

$$\boxed{\partial_x F_z = -\partial_\xi F_x} \;\ldots\; ②$$

$$①, ② \Rightarrow \underbrace{\nabla_\perp F_z}_{\text{accelerating}} = -\partial_\xi \underbrace{\vec{F}_\perp}_{\text{focusing}} \;\ldots\; ③$$

Panofsky-Wenzel theorem for plasma wakefield

In conventional accelerator cavities, for which the original Panofsky-Wenzel theorem was written, multiple modes are supported. In contrast, the plasma wakefield is single mode, so there is no need to integrate over multiple modes to prove this relationship.

Properties of the wakefield:
As usual, we start by writing the the fields in terms of the static and vector potentials:

$$E_z = -\partial_z \phi - \partial A_z / \partial t$$

$$E_z = \frac{\partial \phi}{\partial \xi} - c\frac{\partial A_z}{\partial \xi} \qquad \left(\text{recall } \frac{\partial}{\partial t'} = 0\right)$$

$$= \frac{\partial}{\partial \xi}\left(\phi - cA_z\right)$$

$$\boxed{E_z = \partial_\xi \psi} \quad \ldots (4)$$

Note that the relation we just derived is gauge invariant.

Similarly for the transverse force,

$$F_x = q\left(E_x - c\underbrace{B_y}_{\vec{\nabla}\times\vec{A}}\right)$$

$$= q\left(-\partial_x \phi - \partial_t A_x - c\,\partial_z A_x + c\,\partial_x A_z\right)$$

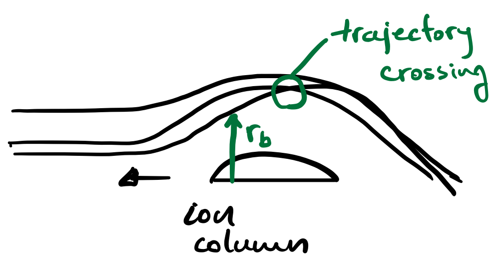
$$= q\left(-\partial_x(\phi - A_z c) - \underbrace{(\partial_t + c\,\partial_z)}_{\equiv \frac{\partial}{\partial t'} \to 0 \text{ (quasistatic approx.)}} A_x\right)$$

$$\left.\begin{array}{ll} \therefore & F_x = -q\,\partial_x \psi \\ \text{Similarly,} & F_y = -q\,\partial_y \psi \end{array}\right\} \Rightarrow \boxed{\vec{F}_\perp = -q\,\nabla_\perp \psi} \quad \ldots (5)$$

Equations 4 and 5 state that the forces are resulting from the same wake potential! Incidentally in this case, the Panofsky-Wenzel theorem is automatically implied. In deriving this relationship we used nothing more than the knowledge of Maxwell's equations and the quasi-static approximation for a wake moving at the speed of light, so it applies to all wakefields whether linear or nonlinear, driven by a beam or laser, whether structured or uniform.

Now the problem becomes how can we find ψ? What we are going to do is to use a phenomenological model of the wakefield based on the observations in simulations:

There are a lot of complicated physics details in the sheath such as trajectory crossing, etc. but the basics of the problem are simpler: we have an ion column, and can define a bubble radius r_b



Our goal is to get an expression for the wake potential as a function of the structure of the ion column and the sheath. This will allow us to get the structure of the forces. To get these forces, we will concentrate on the trajectory of a "particle" along the inner radius of the sheath trajectory. We start with potentials in the Lorentz Gauge

$$\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \end{cases} \quad \cdots (6)$$

From the "Single Particle" Lecture, we know that in the co-moving coordinate under the quasistatic approximation $\left(\frac{\partial}{\partial t'} \to 0\right)$, wave operator $\to -\nabla_\perp^2$

$$6 \Rightarrow \begin{cases} \nabla_\perp^2\phi = -\frac{\rho}{\epsilon_0} \\ \nabla_\perp^2\vec{A} = -\mu_0\vec{J} \end{cases} \quad \cdots (7) \quad \leftarrow \text{in co-moving coordinate}$$

Gauge Condition $\Rightarrow$ $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \quad \cdots (8)$ $\leftarrow$ in normal cartesian coordinates

Rewrite (8) in co-moving coordinate

$$(8) \Rightarrow \vec{\nabla}_\perp\cdot\vec{A}_\perp + \underbrace{\frac{\partial A_z}{\partial z}}_{= -\partial_\xi A_z} + \underbrace{\frac{1}{c}\frac{\partial\phi}{\partial t}}_{= \partial_\xi\phi} = 0 \qquad (\partial_t' = 0)$$

$$\Rightarrow \boxed{\partial_\xi\psi + \vec{\nabla}_\perp\cdot\vec{A}_\perp = 0} \quad \cdots (9)$$

Equation 7 says that the electrodynamic potentials in the co-moving coordinate follow the form of a 2D Poisson equation, and in reaching the answer, we can borrow all of our intuition from 2D electrostatic by pretending that the source term is a charge distribution in 2D with a uniform third dimension.

From here on, we are going to work in <u>normalized</u> units. The two source terms in equation 7 are related through the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

In terms of co-moving coordinates & in normalized units,

$$\frac{\partial}{\partial \xi}(\rho - J_z) + \vec{\nabla}_\perp \cdot \vec{J}_\perp = 0 \quad \ldots (10)$$

Because of the 2D Poisson equations, we can imagine that what happens in each slice is somehow independent than the other slices. Eqn 10 is the continuity equation in that world. So if we integrate over all space in this equation (i.e. in 2D), the second term drops out from Gauss's law evaluated over the boundary of infinite space, leaving the first term as a new conserved quantity, I.e.

$$\int \frac{\partial}{\partial \xi}(\rho - J_z)\, dx_\perp + \underbrace{\int \vec{\nabla}_\perp \cdot \vec{J}_\perp\, dx_\perp}_{\displaystyle \oint_{\infty \text{ boundary}} \vec{J}_\perp \cdot d\vec{l}_\perp = 0} = 0$$

$$\Rightarrow \boxed{\frac{d}{d\xi} \int (\rho - J_z)\, dx_\perp = 0} \quad \ldots (11)$$

The integral over all space in $x_\perp$ can depend only on ξ

This is analogous to total charge in a regular problem, which is conserved. In this problem now, it is the $\int (\rho - J_z) dx_\perp$ that is conserved from one slice to the next.

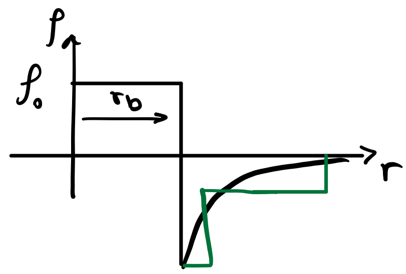
So now because this quantity is conserved, we can develop a model that has this quantity as its source term, and we do so by combining the two terms from equation 7, which in normalized units are expressed as:

$$\left.\begin{aligned} \nabla_\perp^2 \phi &= -\rho \\ \nabla_\perp^2 \vec{A} &= -\vec{J} \end{aligned}\right\} \Rightarrow \nabla_\perp^2 (\underbrace{\phi - A_z}_{\psi \text{ in normalized units}}) = -(\rho - J_z)$$

$$\therefore \boxed{\nabla_\perp^2 \psi = -(\rho - J_z)} \quad \ldots (12)$$

To solve for the potential, we are going to need to know the source term $\rho - J_z$. Note than even the potential inside the ion column depends on these terms outside because from Eqn 11, it is the whole integrated term that is conserved. So now, let's talk about the model for these source terms.

If it was the charge in a slice that was conserved, we could make a model where the electrons from the ion column would have been blown out all the way to the edge and piled up at the sheath, i.e.



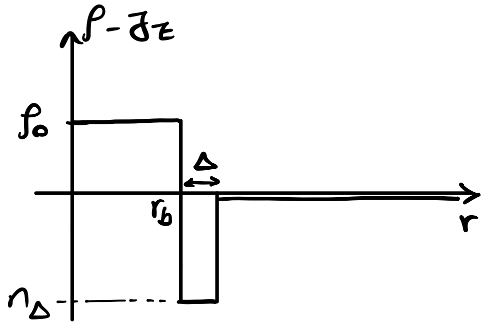
We might even simplify this model by considering a narrow sheath plus a plateau as shown in green:
In that case, the integral under the green and black curve would be the same and would equal to the displaced charge in the ion column.

In an analogous way, we can construct a model for $\rho - J_z$

Inside ion column, $\rho - J_z = \rho_0$ (there are no e^- & ions are assumed immobile)

Outside the bubble, $\rho - J_z = 0$ (undisturbed plasma)

Model the sheath as $\rho - J_z = n_\Delta$



$$\int_0^\infty (\rho - J_z)\, dx_\perp = \text{constant} = 0$$

right in front of wake

$$\int_0^\infty (\rho - J_z)\, 2\pi r\, dr = 0$$

$$\Rightarrow n_\Delta = \frac{r_b^2}{(r_b + \Delta)^2 - r_b^2} \quad \dots \quad (13)$$

So now, to solve our 2D Poisson equations, we could use our electrostatic intuition. Suppose we have the following situation

$-\nabla^2 f = S$ w/ boundary condition that $f \to 0$ as $r \to \infty$, which is the case for ϕ, A, & ψ

What is the scenario where only transverse Laplacian exists, i.e. $\nabla^2 \to \nabla_\perp^2$?
One common case where $\frac{\partial}{\partial z} \to 0$ is a charge that is uniformly distributed in z. Even more specific to our case is one w/ azimuthal symmetry.
For such an ∞ cylinder, we know from electrostatics that $\vec{E} = E_r \hat{r}$ (see e.g. Griffith's "introduction to electrodynamics")
We can use Gauss's Law to find this E_r

$$E_r = -\nabla f$$

$$= \frac{1}{2\pi r} \int_0^r dr' \, 2\pi r' \, S(r')$$

charge density of the ∞ cylinder

$$\therefore f_2 - f_1 = -\int_1^2 \vec{E} \cdot d\vec{l}$$

Choose $d\vec{l} = dr \hat{r}$ since $\vec{E} = E_r \hat{r}$

$$\Rightarrow f_2 - f_1 = -\int_1^2 dr' \cdot \frac{1}{r'} \int_0^{r'} dr'' \, r'' \, S(r'') \quad \ldots (14)$$

Also note that since $r \to \infty$ is our reference for $f = 0$

$$f(r) = f(r) - f(\infty) = -\int_\infty^r [\quad] = -\int_\infty^0 [\quad] + \int_0^r [\quad] \quad \ldots (15)$$

Now, let's look at the two terms in Eqn 15. The first term is integrated over all transverse space, so it could only be a function of ξ. Let's call it $f_0(\xi)$

The beauty of the second term is that the integral only goes up to point "r", the point of observation. This means that for $r < r_b$, the blowout radius, we only need the source term within the ion column, which has a simple form.

So, let's see what Eqn 15 implies about each of the potentials:

For ϕ, $S(r') = \text{constant} = \rho_0 = n_0 e = 1$ in normalized units

$$\Rightarrow \phi = \phi_0(\xi) - \frac{r^2}{4} \qquad r < r_b \quad \text{--} (16)$$

For A_z, $S(r') = J_z$, which is equal to zero inside the bubble

$$\Rightarrow A_z = A_0(\xi) \qquad r<r_b \quad \ldots (17) \qquad 15 \Rightarrow f(r) = f_0(\xi) - \frac{r^2}{4}$$

$$\psi = \phi - A_z = \boxed{\psi_0(\xi) - \frac{r^2}{4}} \quad \ldots (18)$$

The last unknown component of the potentials is $A_\perp$, which can be derived from Eqn 9:

$$\partial_\xi \psi + \vec{\nabla}_\perp \cdot \vec{A}_\perp = 0 \Rightarrow \left(\frac{1}{r}\frac{\partial}{\partial r} r A_r\right) + \underbrace{\partial_\xi \psi}_{\equiv \frac{d}{d\xi}\psi_0} = 0$$

$$\text{Integrating} \Rightarrow \boxed{A_r = -\frac{1}{2} r \frac{d\psi_0}{d\xi}} \quad \ldots (19)$$

Using the n_Δ as the source term for the potential, one can show (HW problem):

$$\psi_0 = (1+\beta)\frac{r_b^2}{4} \quad \ldots (20)$$

$$\beta = \frac{(1+\alpha)^2 \ln(1+\alpha^2)}{(1+\alpha)^2 - 1} \quad \ldots (21)$$

$$\alpha = \frac{\Delta}{r_b} \quad \ldots (22)$$

Note that the equations 20-22 are dependent on our choice of sheath model. Here, we discuss a very simple model, which can be refined to give even more accurate results as needed (see e.g. T.N. Dalichaouch, PoP 28, 063103 (2021)).

From the potentials, we can now proceed to find the fields and the forces within the wakefield. We still don't know $r_b(\xi)$, so we will have to get to that later as well. In regular coordinates,

$$\begin{cases} \vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{E}_\perp = \hat{r}\left(-\dfrac{\partial\phi}{\partial r} - \partial_\xi A_r\right) & \ldots (23) \\ E_z = \partial_\xi \psi & \text{(see Eqn. 4)} \end{cases}$$

$$\begin{cases} B_\phi = \partial_z A_r - \partial_r A_z \\ B_r = \dfrac{1}{r}\left(\partial_\phi A_z - r\,\partial_z A_\phi\right) \\ B_z = \dfrac{1}{r}\left(\partial_r r A_\phi - r\,\partial_\phi A_r\right) \end{cases}$$

Because of Azimuthal symmetry, there is no variation in ϕ & no azimuthal currents (particles are blown outward from an e-beam or a laser). Therefore, there is no source term J_ϕ & so $A_\phi = 0$

Moreover, since $A_z = A_0(\xi)$, $\partial_r A_z = 0$

$$\left.\begin{aligned} &\Rightarrow \quad B_r = 0 \quad \& \quad B_z = 0 \\ &\Rightarrow \quad \vec{B} = B_\phi \hat{\phi} = \partial_z A_r \end{aligned}\right\} \cdots (24)$$

Forces

In the co-moving coordinate, then, the transverse force, $F_\perp$ is given by

$$\vec{F_\perp} = q(E_r - V_z B_\phi)\,\hat{r}$$

$$= \hat{r}\, q\left(-\frac{\partial\phi}{\partial r} - \partial_\xi A_r + V_z \partial_\xi A_r\right)$$

$$\boxed{\vec{F_\perp} = \hat{r} q\left(-\frac{\partial\phi}{\partial r} - (1 - V_z)\partial_\xi A_r\right)} \cdots (25)$$

Note that this is the field inside the wake. For a particle traveling at the speed of light, e.g. that of an electron being accelerated,

$$V_z \sim c = 1 \text{ in normalized units}$$

$$\left.\begin{aligned} &\Rightarrow F_\perp = -\hat{r} q \frac{\partial}{\partial r}\phi \\ &\phi = \phi_0(\xi) - \frac{r^2}{4} \end{aligned}\right\} \Rightarrow \boxed{F_\perp = -\hat{r}\frac{r}{2}} \cdots (26)$$

Note that the focusing force in this cavity is radially inward and linear and does not depend on ξ. From the Panofsky-Wenzel the accelerating force in this cavity does not depend on 'r'. These are great properties for accelerating electrons. Now that we have the transverse force, we can write the transverse equation of motion for a particle in the sheath and derive an expression for r_b, on which the expressions for the potentials depend. In regular coordinates,

$$\frac{d}{dt} P_\perp = \frac{d}{dt}\left[\gamma V_r(r_b)\right] = F_\perp(r_b)$$

In the co-moving coordinates,

$$25 \longrightarrow \quad RHS = \partial_r \phi \Big|_{r_b} + (1-V_z)\,\partial_\xi A\Big|_{r_b}$$

$$16 \rightarrow \quad \frac{\partial \phi}{\partial r}\Big|_{r_b} = -\frac{r_b}{2}$$

$$19 \rightarrow \quad \frac{\partial A_r}{\partial \xi}\Big|_{r_b} = -\frac{1}{2} r_b \frac{d^2 \psi_0}{d\xi^2}$$

$$\Rightarrow \quad RHS = -\frac{r_b}{2} - (1-V_z)\frac{1}{2} r_b \frac{d^2\psi_0}{d\xi^2} \quad \cdots \text{(27)}$$

For LHS, we use $\frac{d}{dt} = (1-V_z)\frac{d}{d\xi}$ (see single-particle lecture)

$$\frac{d}{dt}\left[\gamma V_r(r_b)\right] = \frac{d}{dt}\left[\gamma \frac{d}{dt} r_b\right]$$

$$= (1-V_z)\frac{d}{d\xi}\gamma(1-V_z)\frac{d}{d\xi} r_b \quad \cdots \text{(28)}$$

Recall, the constant of motion (in SI units) is given by

$$\frac{d}{dt}\left[\gamma mc^2 - P_z c + q(\phi - A_z c)\right] = 0$$

For an e^-, $q = -e$. We divide by mc^2 to get to the normalized units:

$$\frac{d}{dt}\left[\gamma - \frac{P_z}{mc} - \frac{e}{mc^2}\psi\right] = 0$$

$$\Big\downarrow \text{normalized units}$$

$$\frac{d}{dt}\left[\gamma - P_z - \psi\right] = 0$$

$$\Rightarrow \quad \gamma - P_z - \psi = C$$

An e^- in the sheath that start in front of the driver is initially at rest $\Rightarrow \quad \gamma = 1, \quad \psi = 0, \quad P_z = 0$

$$\Rightarrow \quad C_e = 1$$

$$\Rightarrow \boxed{\gamma - P_z = 1+\psi} \quad \cdots (29)$$

Now, $P_z = \gamma v$ in normalized units

$$29 \rightarrow \quad \gamma(1 - v_z) = 1+\psi \quad \cdots (30)$$

substitute (30) in (28)

$$\Rightarrow \text{LHS} = (1-v_z)\frac{d}{d\xi}(1+\psi)\frac{d}{d\xi}r_b \cdots (31)$$

$$\text{From } 27, \; \text{RHS} = -\frac{r_b}{2} - (1-v_z)\frac{1}{2}r_b\frac{d^2\psi_0}{d\xi^2}$$

There are 3 variables in this eqn: r_b, v_z, & ψ. If we can substitute all variables in terms of r_b, we will get a single differential equation:

$$20 \rightarrow \psi = \left[1+\beta(r_b)\right]\frac{r_b^2}{4} - \frac{r^2}{4}$$

for $r = r_b$ (at the sheath) $\Rightarrow \boxed{\psi(r_b) = \beta\frac{r_b^2}{4}} \quad \cdots (32)$

For v_z, we use the relation in constant of motion:

$$\left.\begin{array}{l} 29 \rightarrow \gamma - P_z = 1+\psi \\ \gamma^2 = 1+p^2 \Rightarrow \gamma = \sqrt{1+P_z^2+P_\perp^2} \end{array}\right\} \Rightarrow \sqrt{1+P_z^2+P_\perp^2} = 1+\psi+P_z$$

$$\Rightarrow 1+\cancel{P_z^2}+P_\perp^2 = (1+\psi)^2 + 2(1+\psi)P_z + \cancel{P_z^2}$$

$$\Rightarrow \boxed{P_z = \frac{1+P_\perp^2-(1+\psi)^2}{2(1+\psi)}} \quad \cdots (33)$$

$$29 \rightarrow \gamma = (1+\psi)+P_z \;\Rightarrow\; \gamma = \frac{1+P_\perp^2+(1+\psi)^2}{2(1+\psi)} \cdots (34)$$

For $1-v_z$, use 29 again, $\gamma - P_z = \gamma(1-v_z) = 1+\psi$

$$\Rightarrow \frac{1}{1-V_z} = \frac{\gamma}{1+\psi} = \frac{1+P_\perp^2+(1+\psi)^2}{2(1+\psi)^2} \quad \cdots (35)$$

Now that we have an expression for $(1-V_z)^{-1}$, the only variable that doesn't depend on r_b is $\frac{P_\perp^2}{(1+\psi)^2}$

Use $P_\perp = \gamma \frac{d}{dt} r_b \Rightarrow \frac{P_\perp^2}{(1+\psi)^2} = \left[\frac{\gamma\, d/dt\, r_b}{1+\psi}\right]^2 = \left[\cancel{\frac{\gamma(1-V_z)}{1+\psi}} \frac{dr_b}{d\xi}\right]^2$

Cancel from constant of motion, Eqn. 29

$$\Rightarrow \frac{P_\perp^2}{(1+\psi)^2} = \left[\frac{dr_b}{d\xi}\right]^2 \quad \cdots (36)$$

$$35 \,\&\, 36 \Rightarrow \frac{1}{1-V_z} = \frac{1}{2}\left[1 + \frac{1}{(1+\psi)^2} + \left(\frac{dr_b}{d\xi}\right)^2\right] \quad \cdots (37)$$

Divide the equation of motion (Eqn. 27 & 31) by $1-V_z$

$$\Rightarrow \frac{d}{d\xi}\left[(1+\psi(r_b))\frac{dr_b}{d\xi}\right] = \frac{-r_b}{2(1-V_z)} - \frac{1}{2} r_b \frac{d^2\psi_0}{d\xi^2}$$

substitute 37,

$$\boxed{\frac{d}{d\xi}\left[(1+\psi(r_b))\frac{dr_b}{d\xi}\right] = -\frac{r_b}{4}\left[1 + \left(\frac{dr_b}{d\xi}\right)^2 + \frac{1}{(1+\psi(r_b))^2}\right] - \frac{r_b}{2}\frac{d^2\psi_0(r_b)}{d\xi^2} \quad \cdots (38)}$$

where

$$\psi(r_b) = \beta\left(\frac{\Delta}{r_b}\right)\frac{r_b^2}{4} \qquad \psi_0 = \left(1+\beta\left(\frac{\Delta}{r_b}\right)\frac{r_b^2}{4}\right)$$

$$\beta = \frac{(1+\alpha)^2 \ln(1+\alpha)^2}{(1+\alpha)^2 - 1} - 1 \qquad \alpha \equiv \frac{\Delta}{r_b}$$

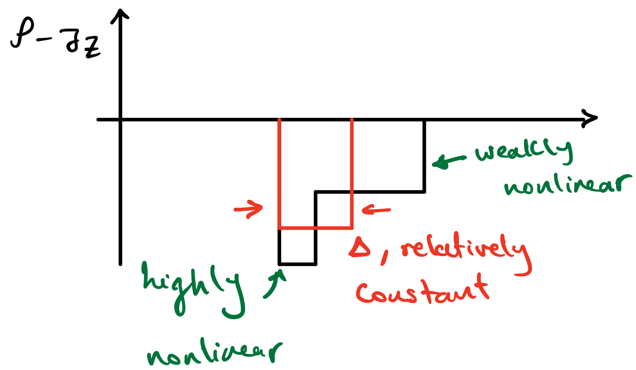
These expressions derived by Wei Lu in 2006 describe the fields with the simplest phenomenological model as we could conjure up!

Consider the case of very strong blowout, which occurs when the energy density of the driver is very high ($a_0 \gg 1$ for a laser & $\Lambda \gg 1$ for a particle beam). In that case,

$r_b \gg 1$

Two insights for this regime come from simulations:

1. Δ/r_b is relatively constant & on the order of 0.3 or smaller.
2. The sheath is not a uniform section as we assumed. In fact it has a narrow high density part and a larger lower density part, where the motion of electrons is weakly nonlinear. Nevertheless, we make the assumption that we can lump them into a single section:



$\therefore$ $\alpha \approx$ small

If you Taylor expand $\ln$ in the expression for β,

$$\beta(\Delta, r_b) \approx \alpha \quad , \quad \text{small}$$

Expressing Eqn 38 only in terms of β & r_b & simplifying using small α & β, the differential equation simplifies to

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = 0 \quad \dots \quad (39)$$

Comparing with the equation for a circle,

$$x^2 + y^2 = 1 \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0 \quad \dots \quad (40)$$

We see that eqn 39 is off from the eqn of a circle only by the coefficient of middle term. Where this term is small, e.g. near the top of the bubble, eqn. 39 & 40 merge & at that location, the ion column is almost a sphere.

Now that we have an equation for the blowout radius, we can find an expression for ψ and for E_z

Inside the bubble, $\psi = (1+\beta)\frac{r_b^2}{4} - \frac{r^2}{4}$ $\quad$ $r_b \gg 1 \rightarrow \beta \ll 1$

$$\psi \approx \frac{r_b^2}{4} - \frac{r^2}{4} \quad \cdots \quad (41)$$

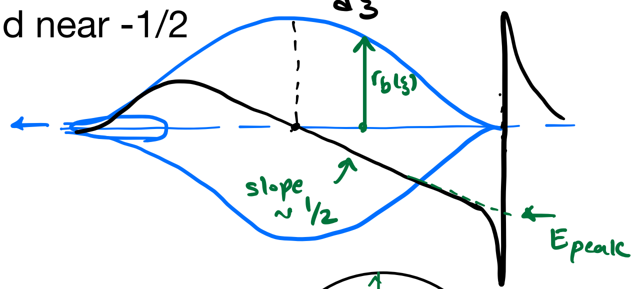
$$4 \rightarrow \quad E_z = \frac{\partial \psi}{\partial \xi} \approx \frac{r_b}{2}\frac{dr_b}{d\xi} \quad \cdots \quad (42)$$

$$\Rightarrow \frac{dE_z}{d\xi} = \frac{1}{2}\left(\frac{dr_b}{d\xi}\right)^2 + \frac{1}{2} r_b \frac{d^2 r_b}{d\xi^2}$$

$$\text{using Eqn 39} \rightarrow \frac{dE_z}{d\xi} = -\frac{1}{2}\left[1 + \left(\frac{dr_b}{d\xi}\right)^2\right] \quad \cdots \quad (43)$$

Equation 43 implies that near the top of the bubble, where $\frac{dr_b}{d\xi} \approx 0$, the slope of the accelerating field is linear and near -1/2

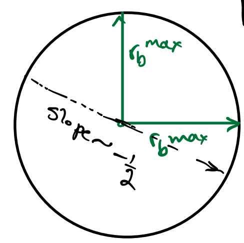
The full field looks something like this:



Max field predicted assuming slope of $-1/2$:

$$E_{peak} = r_{b,max} \times \text{slope} = -\frac{r_{bmax}}{2}$$

sometimes called useful field



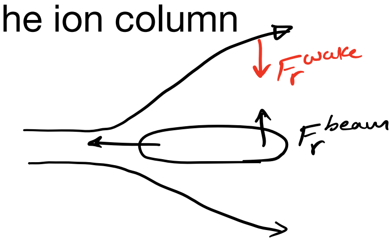
$F_\perp = -\hat{r}\,\frac{r}{2} \Rightarrow$ linear focusing force

$\Rightarrow$ Limited or controlled emittance growth

Note that x & y forces are coupled.

Now, to find the maximum blowout radius, $r_{b,max}$, Wei Lu's paper in 2006 used a source term for the differential equation in terms of beam charge per unit length, λ, but for a large range of parameter, you can estimate what size blowout you have using a back of the envelope calculation. The great insight in this paper was to realize that it is the trajectory of the innermost electron along with some phenomelogical model that determines many of the properties of the wake.

We can estimate the size of the bubble by looking for an equilibrium radius, here the field of the drive beam balances that of the ion column



At each slice, for equilibrium radius,

$$F_{r,w} + F_{r,b} = 0 \Rightarrow r_{eq}$$

It turns out that

$$r_{b\max} \sim \underbrace{1-2\, r_{eq}}_{\sim \sqrt{2} \text{ for a wide range of parameters.}}$$

in normalized units,

$$\left.\begin{aligned} F_{r,w} &= \frac{r}{2} \\ F_{r,b} &= \frac{\partial \phi}{\partial r} = \frac{\lambda(\xi)}{r} \end{aligned}\right\} \Rightarrow \frac{r_{eq}}{2} = \frac{\lambda}{r_{eq}} \Rightarrow \boxed{r_{eq} = \sqrt{2}\sqrt{\lambda}} \quad \ldots (44)$$

↑ for each slice

↑ recall from single particle motion notes, where $\lambda(\xi) = \int_0^{r_b} \rho(r)\, r\, dr$

$$r_{b,max} \sim \sqrt{2}\, r_{eq} = 2\sqrt{\lambda} \quad \ldots (45)$$

Finally,

$$F_{z,max} = \frac{r_{b,max}}{2} = \sqrt{\lambda(\xi_p)} ,$$

where ξ_p is the location of peak beam charge per unit length.