So far, we have investigated the properties of plasma structures in 1D and 3D. In this section, we want to develop a self-consistent formalism for how a driver, i.e. a laser or beam generates the plasma wave and how the plasma wave in turn modifies the driver. There are some analogies between a laser and particle beam driver and there are some important differences, as we will investigate later in the context of linear theory.

We will see that the coupling between the laser and the wake is given by two coupled differential equations, which constitute the most important equations in short-pulse laser plasma interactions:

$$\begin{cases} 2\partial_s \partial_\xi A - \nabla_\perp^2 A + \dfrac{1}{1+\psi} A = 0 \\ \partial_\xi^2 \psi + \dfrac{1}{2}\left[1 - \dfrac{1+A^2}{(1+\psi)^2}\right] = 0 \end{cases} \quad \ldots (1)$$

We will start from the relativistic cold fluid plasma equations to derive these equations and then make approximations to study them in the linear limit.

Euler's eqn for e^-: $$\underbrace{\left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)}P = -e\left(\vec{E} + \vec{V}\times\vec{B}\right) \quad \ldots (2)$$

(Recall: ions are fixed & immobile)

Convective derivative $= \dfrac{D}{D_t}\vec{P}$

Continuity eqn: $$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{V}) = 0 \quad \ldots (3)$$

Note that since we are in the cold fluid limit, the "fluid velocity" is the same as the velocities of all the particles in the fluid element (since there is no thermal spread), so that the total derivative can be interpreted as describing the motion of a single particle.

Now, since the Euler's equation is the same as the equation of motion of a single particle in the E&M field, the equation for canonical momentum can be derived in the same way, which gives:

$$\frac{D}{D_t}\left(\vec{P} - e\vec{A}\right) = -e\left(-\nabla\phi - (\vec{\nabla}\vec{A})\cdot\vec{v}\right) \quad \ldots (4)$$

This equation is consistent with Hamiltonian mechanics, where we know that if there is translational invariant in any direction (i.e. where the gradient term is zero), then the canonical momentum in that direction is conserved.

Since the canonical momentum is not conserved in general, we are going to look for another conservation condition for the canonical momentum in plasma, and we find that by taking the curl of the Euler equation

$$\vec{\nabla}\times ② \Rightarrow \partial_t \vec{\nabla}\times\vec{P} + \vec{\nabla}\times(\vec{v}\cdot\vec{\nabla})P = -e\left[\vec{\nabla}\times\vec{E} + \vec{\nabla}\times(\vec{v}\times\vec{B})\right]$$

from vector identities,

$$(\vec{v}\cdot\vec{\nabla})\vec{P} = -\vec{v}\times\vec{\nabla}\times\vec{P} - \underbrace{(\vec{\nabla}\vec{P})\vec{v}}_{=\frac{1}{\gamma}P_j\partial_i P_j = mc^2\partial_i\gamma \text{ (exercise!)}} \quad \ldots ⑤$$

$$\Rightarrow \partial_t \vec{\nabla}\times\vec{P} + \vec{\nabla}\times\left(-mc^2\nabla\gamma - \vec{v}\times\vec{\nabla}\times\vec{P}\right) = -e\left[-\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{v}\times\vec{B}\right]$$

$\vec{\nabla}\times(\vec{\nabla}\ )=0$ Faraday's law

$$\Rightarrow \partial_t\left(\vec{\nabla}\times\vec{P} - e\vec{B}\right) - \vec{\nabla}\times\vec{v}\times\left(\vec{\nabla}\times\vec{P} - e\vec{B}\right)$$

$\vec{\nabla}\times\vec{A}$

Define vorticity $\Lambda = \vec{\nabla}\times\vec{\mathbb{P}} = \vec{\nabla}\times(\vec{P} - e\vec{A})$

canonical momentum

$$\boxed{\frac{\partial}{\partial t}\Lambda + \vec{\nabla}\times\vec{v}\times\vec{\Lambda} = 0} \quad \ldots ⑥$$

For example, consider a case where a laser pulse comes through a plasma. Before the arrival of the laser pulse, the vorticity is zero, which implies its initial time derivatives are zero

$\Lambda = 0$ initially everywhere; time rate of change $= 0$

$$\vec{P}\ \&\ \vec{A} = 0 \Rightarrow \vec{\nabla}\times\vec{v}\times\vec{\Lambda} = 0 \Rightarrow \partial_t\vec{\Lambda} = 0$$

$$\Rightarrow \Lambda \equiv 0 \quad \text{(remains zero for all time)} \quad \ldots ⑦$$

Now, we can use the knowledge of the value of vorticity to get a relationship between the magnetic field and the curl of the fluid momentum in the plasma:

$$\vec{\nabla}\times(\vec{p} - e\vec{A}) = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{p} = e\vec{B}} \quad \ldots ⑧$$

$$\therefore \text{Euler's equation: } \frac{\partial\vec{P}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{P} = -e(\vec{E} + \vec{v}\times\vec{B})$$

$$\therefore \frac{\partial \vec{p}}{\partial t} = -e\vec{E} - \vec{v}\times\vec{\nabla}\times\vec{p} - (\vec{v}\cdot\vec{\nabla})\vec{p}$$

eqn 5 above

$$\therefore \boxed{\frac{\partial \vec{p}}{\partial t} = -e\vec{E} - mc^2\nabla\gamma} \quad \cdots (9)$$

This is an alternative form for Euler's equation in an initially unmagnetized plasma.

Note that the time derivative of the fluid momentum is a partial derivative. The grad term on the right hand side is the term that allows the laser to put radiation pressure on the plasma and is the source term in the fluid view of the ponderomotive force. We will see that later in the linear limit, but it is also possible to derive that in the nonlinear limit.

Where do we go from here? There are several ways of proceeding: we can for example use these relations to find an expression for the wake function in the plasma. But before then, it will be instructive for us to find a single equation for the fluid momentum of the plasma. The vorticity then would give the coupling between the vector potential of the driver (e.g. laser) and the momentum.

<u>Getting a single equation for momentum:</u>

We already know that we are looking for something like a wave response. So, we take the time derivative of equation 9 to see if we can get a wave equation for the plasma fluid momentum:

$$\partial_t (9) \Longrightarrow \quad \partial_t^2 \vec{p} = -e\,\underbrace{\partial_t \vec{E}}_{\uparrow} - mc^2\nabla\partial_t\gamma \quad \cdots (10)$$

from Ampere's Law: $(\vec{\nabla}\times\vec{B} - \mu_0\vec{J})\cdot c^2$

current is due to $e^- \Rightarrow \quad \vec{J} = -en\vec{v}$

in terms of momentum $\Rightarrow \quad = -e\underbrace{\left(\frac{n}{\gamma}\right)}_{\uparrow}\vec{p}$

called proper density

Next is relating the magnetic field, n, and γ to the momentum. We already know from vorticity:

$$\vec{B} = e(\vec{\nabla}\times\vec{p})$$

$$\gamma = \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}$$

For n, use Gauss's law & Euler's eqn:

$$\underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{replace using Euler's eqn}} = \frac{e}{\epsilon_0}\,(\underbrace{n_0}_{\text{ion density}} - \underbrace{n}_{e^- \text{ density}})$$

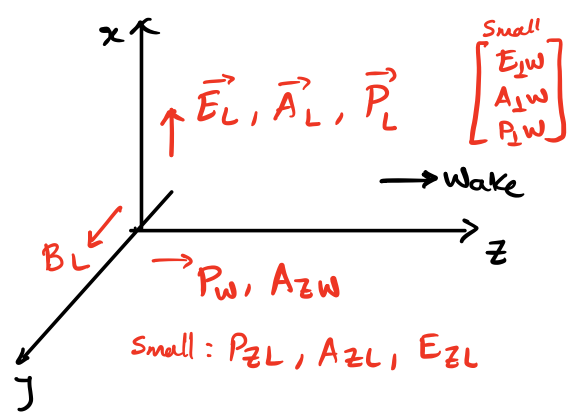
$$\partial_t \vec{P} = -e\vec{E} - mc^2\nabla\gamma \Rightarrow \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\left\{\frac{1}{e}\left(\partial_t\vec{P} + mc^2\nabla\gamma\right)\right\}$$

Substituting these terms in eqn 10 results in

$$\partial_t^2 P + c^2\,\vec{\nabla}\times\vec{\nabla}\times\vec{P} + \underbrace{\frac{1}{\gamma}\left[\omega_{p_0}^2 + \frac{1}{m}\partial_t\vec{\nabla}\cdot\vec{P} + c^2\nabla^2\gamma\right]}_{\text{This is } \frac{n}{\gamma}}\vec{P} + mc^2\partial_t\nabla\gamma = 0 \quad \ldots (11)$$

Now, let's consider our specific problem: the wake has a component that is due to the laser and another that is given by the wake:

$$\vec{P} = \vec{P}_{Laser} + \vec{P}_{wake}$$



We note that for a laser with finite size, there are components of the field that are in the 'z' direction (since $\vec{\nabla}\cdot\vec{E}=0$), and they scale roughly by the inverse of laser wave number to the plasma frequency:

$$\vec{\nabla} = \hat{z}\,\partial_z + \vec{\nabla}_\perp$$

$$\partial_z \sim k_0$$

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow E_z \sim k_0^{-1}\nabla_\perp$$

$$\sim \frac{\omega_p}{\omega_0} \equiv \underbrace{\epsilon}_{\text{smallness parameter}} \ll 1$$

Similarly the transverse field of the laser is larger than that of the wake by the inverse of this number. So without going through a rigorous derivation, we are making an ordering that the transverse momentum is dominated by the force of the laser, while the longitudinal momentum is dominated by the wake. With that in mind, we are going to split the momentum equation into longitudinal and transverse equations and see if we can separate these two effects:

$$\hat{z}: \partial_t^2 P_z \underbrace{- c^2 \nabla^2 P_z + c^2 \partial_z \vec{\nabla}\cdot\vec{P}}_{\text{use } \vec{\nabla}\times\vec{\nabla}\times\vec{P} = \vec{\nabla}(\vec{\nabla}\cdot\vec{P}) - \nabla^2 P} + \left[\omega_{p_0}^2 + \frac{1}{m}\partial_t \vec{\nabla}\cdot\vec{P} + c^2\nabla^2\gamma\right]\frac{P_z}{\gamma} + m c^2 \partial_t \partial_z \gamma = 0$$

use $\vec{\nabla} = \partial_z + \nabla_\perp$ to break up Laplacian & $\vec{\nabla}\cdot(\,)$

$$\Rightarrow \partial_t^2 P_z - \cancel{c^2 \partial_z^2 P_z} - c^2 \nabla_\perp^2 P_z + \cancel{c^2 \partial_z^2 P_z} + c^2 \partial_z \vec{\nabla}_\perp \cdot \vec{P}_\perp + \left[\omega_{p_0}^2 + \frac{1}{m}\partial_t \vec{\nabla}\cdot\vec{P} + c^2\nabla^2\gamma\right]\frac{P_z}{\gamma} + m c^2 \partial_t \partial_z \gamma = 0 \quad \ldots (12)$$

for $\hat{x}$: $$\partial_t^2 P_x - c^2 \nabla^2 P_x + c^2 \partial_x \vec{\nabla}\cdot\vec{P} + \left[\omega_{p_0}^2 + \frac{1}{m}\partial_t \vec{\nabla}\cdot\vec{P} + c^2\nabla^2\gamma\right]\frac{P_x}{\gamma} + m c^2 \partial_t \partial_x \gamma = 0 \quad \ldots (13)$$

Now, we transition to normalized units and a co-moving coordinate

$$\frac{\omega_{p_0}}{c}(x,y,z) \to (x,y,z) \qquad \frac{v}{c} \to v \qquad \frac{e}{mc^2}\begin{Bmatrix}\phi \\ c\vec{A}\end{Bmatrix} \to \begin{Bmatrix}\phi \\ \vec{A}\end{Bmatrix}$$

$$\omega_{p_0} t \to t \qquad \frac{P}{mc} \to P$$

$$\frac{n}{n_0} \to n \qquad \frac{e}{mc\omega_{p_0}}\begin{Bmatrix}E \\ cB\end{Bmatrix} \to \begin{Bmatrix}E \\ B\end{Bmatrix}$$

Normalized equations

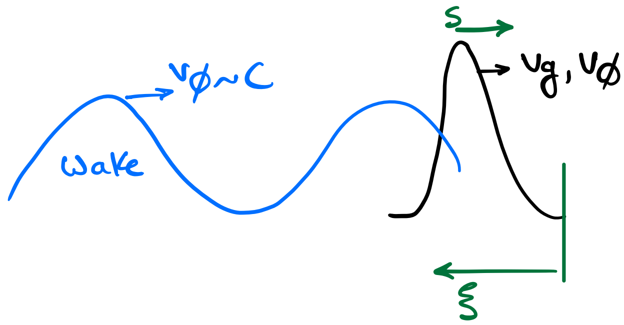
$$\hat{z}: \partial_t^2 P_z - \nabla_\perp^2 P_z + \partial_z \vec{\nabla}_\perp\cdot\vec{P}_\perp + \left[1 + \partial_t \partial_z P_z + \partial_t \vec{\nabla}_\perp\cdot\vec{P}_\perp + \partial_z^2\gamma + \nabla_\perp^2\gamma\right]\frac{P_z}{\gamma} + \partial_t \partial_z \gamma = 0 \quad \ldots (14)$$

$$\hat{x}: \partial_t^2 P_x - \partial_z^2 P_x - \nabla_\perp^2 P_x + \partial_x \vec{\nabla}\cdot\vec{P} + \left[1 + \partial_t \partial_z P_z + \partial_t \vec{\nabla}_\perp\cdot\vec{P}_\perp + \partial_z^2\gamma + \nabla_\perp^2\gamma\right]\frac{P_x}{\gamma} + \partial_t \partial_x \gamma = 0 \quad \ldots (15)$$

Since we are focusing on a short-pulse laser and a wake both of which have phase velocities near the speed of light, we use a co-moving coordinates. Unlike the first class though, here we use

$\vec{x}_\perp = \hat{x}x + \hat{y}y \qquad ct - z = \xi \qquad \& \quad s = z$

Here, 's' represents the distance that a position 'z' of a point of constant ξ (E.g. the front of laser). Since the co-moving coordinate travels at the speed of light, this is equivalent to 'ct', where 't' is time in the original frame. Meanwhile, ξ is still the distance from the front of the laser.



Recall that using 'ct' in ξ instead of $v_\phi t$ introduces an error that is proportional to $1/2\gamma^2$

Additionally, we make the quasi-static approximation once again, which in this case takes the form of
$$\frac{\partial}{\partial s} \ll \frac{\partial}{\partial \xi}$$

The derivatives in this translation are

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial s}\frac{\partial s}{\partial z} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial s} \overset{\text{quasistatic approximation}}{\approx} -\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial s}\cancelto{0}{\frac{\partial s}{\partial t}} = c\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \quad (\text{in normalized units})$$

Now, using the co-moving coordinate and in the normalized units, the equations become

$$\hat{z}:\ \partial_\xi^2 P_z - \nabla_\perp^2 P_z - \partial_\xi \vec{\nabla}_\perp \cdot \vec{P}_\perp + \left[1 - \partial_\xi^2 P_z + \partial_\xi \vec{\nabla}_\perp \cdot \vec{P}_\perp + \partial_\xi^2 \gamma + \nabla_\perp^2 \gamma\right]\frac{P_z}{\gamma} - \partial_\xi^2 \gamma = 0$$

$$\Rightarrow \partial_\xi^2 (P_z - \gamma) - \nabla_\perp^2 P_z - \partial_\xi \vec{\nabla}_\perp \cdot \vec{P}_\perp + \underbrace{\frac{1}{\gamma}\left[1 + \partial_\xi^2(\gamma - P_z) + \partial_\xi \vec{\nabla}_\perp \cdot \vec{P}_\perp + \nabla_\perp^2 \gamma\right]}_{n/\gamma,\ \text{proper density}} P_z = 0 \quad \ldots (16)$$

Recalling that $\gamma - P_z$ is an important component of the constant of motion, we gather these terms (multiply equation above by -1 first):

$$\partial_\xi^2(\gamma - P_z)\left[1 - \frac{P_z}{\gamma}\right] + \nabla_\perp^2 P_z + \partial_\xi \vec{\nabla}_\perp \cdot \vec{P}_z \left[1 - \frac{P_z}{\gamma}\right] - \frac{P_z}{\gamma} - \nabla_\perp^2 \gamma \frac{P_z}{\gamma} = 0$$

- Multiply through by $\frac{1}{1-P_z/\gamma} = \frac{\gamma}{\gamma - P_z}$
- add & subtract the term $\frac{P_z}{\gamma-P_z}\nabla_\perp^2 P_z$
- collect terms to get

$$\boxed{\partial_\xi^2(\gamma-P_z) - \frac{P_z}{\gamma-P_z}\left(1+\nabla_\perp^2(\gamma-P_z)\right) + \partial_\xi \nabla_\perp\cdot\vec{P}_\perp + \nabla_\perp^2 P_z = 0} \quad \cdots (17)$$

This is the equation for the evolution of a fluid element. From the previous lectures, we know that the wake potential ($\psi = \phi - A_z$) is an important quantity from which all the important properties of the wake can be derived. We therefore desire to get an equation for the wake potential. We can do so using the constant of motion for our fluid element, which starts from rest in a field-free region:

$$\gamma - P_z - \psi = 1 \Rightarrow \gamma - P_z = \psi + 1$$

from the 3D nonlinear lecture (Eqn 33 & 34), P_z & $P_\perp$ are related by

$$P_z = \frac{(1+P_\perp^2) - (1+\psi)^2}{2(1+\psi)}$$

$$\& \qquad \gamma = 1+\psi+P_z = \frac{1+P_\perp^2+(1+\psi)^2}{2(1+\psi)}$$

substitute in eqn 17 to get

$$\boxed{\partial_\xi^2\psi + \frac{1}{2}\left[1 - \frac{(1+P_\perp^2)}{(1+\psi)^2}\right] - \frac{(1+P_\perp^2)}{(1+\psi)^2}\nabla_\perp^2\psi - \partial_\xi \nabla_\perp\cdot P_\perp + \frac{\nabla_\perp^2 P_\perp^2}{2(1+\psi)} = 0} \quad \cdots (18)$$

Alternatively, one could realize that the terms in square bracket in equation 16 are equal to $\frac{n}{\gamma}$. Then, using eqn for ψ in the comoving coordinate gives:

$$-\nabla_\perp^2\psi = \rho - J_z = n_0 - n(1-v_z) \quad \text{normalized}$$

$$= 1 - \frac{n}{\gamma}(\gamma - P_z)$$

$$\Rightarrow -\nabla_\perp^2 \psi = 1 - \frac{n}{\gamma}(1+\psi)$$

$$\Rightarrow \frac{n}{\gamma} = \frac{1+\nabla_\perp^2 \psi}{(1+\psi)} \quad \cdots (19)$$

Substituting Eqn 19 for the proper density in Eqn 16 and substituting for γ and P_z as above, results in Eqn 18. This is the alternate way of deriving this equation.

Note that Eqn 18 is a *fully nonlinear* equation for the wake evolution. The $P_\perp$ is primarily provided by the driver, which is why we will leave it here as the source term for ψ. Alternatively, ψ is primarily due to the wake. In this way, Eqn 18 gives us a direct method for coupling the energy of a drive beam to the plasma wakefield.

To simplify the problem, we are going to consider the case of a transversely uniform driver, or a wake with a small transverse gradient:

$$\nabla_\perp \ll \partial_\xi \quad \text{(spotsize is wider than skin depth, } c/\omega_p\text{)}$$

So, we drop the $\nabla_\perp$ terms:

$$\boxed{\partial_\xi^2 \psi + \frac{1}{2}\left[1 - \frac{(1+P_\perp^2)}{(1+\psi)^2}\right] \approx 0} \quad \cdots (19)$$

Eqn 19 describes how a wake is excited by the driver, since, as discussed above, the transverse fluid momentum (the $P_\perp$ term) is dominated by the laser. Therefore, to examine the evolution of the laser, we need to look at the equation in the $\hat{x}$ direction.

Eqn 15: $$\partial_t^2 P_x - \partial_z^2 P_x - \nabla_\perp^2 P_x + \partial_x \vec{\nabla}\cdot\vec{P} + \left[1 + \partial_t \partial_z P_z + \partial_t \vec{\nabla}_\perp \cdot \vec{P}_\perp + \partial_z^2 \gamma + \nabla_\perp^2 \gamma\right]\frac{P_x}{\gamma} + \partial_t \partial_x \gamma = 0$$

In writing this equation in the co-moving coordinate, we note that the first two terms will cancel out, leaving no terms to describe the longitudinal evolution of P_x. To alleviate this problem, we keep the next order leading term in the ∂_z^2 expression:

$$\partial_z^2 = \partial_s^2 + \partial_\xi^2 - 2\partial_s\partial_\xi$$

In our quasistatic ordering, $\partial_\xi^2 \gg \partial_s\partial_\xi \gg \partial_s^2$

$$\Rightarrow \hat{x}: \cancel{\partial_\xi^2 P_x} - \cancel{\partial_\xi^2 P_x} + 2\partial_\xi\partial_s P_x - \nabla_\perp^2 P_x + \partial_x \vec{\nabla}\cdot\vec{P} + \underbrace{\left(\frac{n}{\gamma}\right)}_{= \frac{1+\nabla_\perp^2\psi}{1+\psi}} P_x$$

$$+ \partial_\xi \partial_x \gamma = 0 \quad \ldots (20)$$

This equation has contributions from wake and the laser. Since we know that the transverse momentum is dominated by the laser, we use our knowledge of the laser quantities to further simplify this equation:

Laser wave in plasma is transverse wave $(\vec{k} \perp \vec{E})$

$$\Rightarrow \vec{\nabla}\cdot\vec{P} \sim \vec{\nabla}\cdot\vec{E} \sim \vec{k}\cdot\vec{E} = 0$$

Next, we drop the terms that are small for a motion oscillating on the order of laser freq. (ω_0). For such motion, transverse derivatives are generally small $(\sim k_0^{-1}\nabla)$, so we drop $\partial_\xi\partial_x\gamma$ & $\nabla_\perp^2\psi$, but we keep $\nabla_\perp^2 P_x$, since that is the term that gives rise to diffraction.

With this approximation,

$$2\partial_\xi\partial_s P_x - \nabla_\perp^2 P_x + \frac{1}{1+\psi} P_x = 0 \quad \ldots (21)$$

We also note that since $\nabla_\perp \sim 0$, the transverse canonical momentum is conserved:

Eqn 4 → (normalized) $\frac{D}{Dt}(\vec{P} - \vec{A}) = 0$ ⇒ for a particle starting from rest, in front of the laser, this constant is zero

$$\Rightarrow \vec{P} = \vec{A}$$

$$\boxed{\vec{P_\perp} = \vec{A_\perp}} \quad \cdots (22)$$

with this last substitution, we recover eqn 1 from eqn 19 & 21:

$$\begin{cases} 2\,\partial_\xi \partial_s A - \nabla_\perp^2 A + \frac{1}{1+\psi} A = 0 \\ \partial_\xi^2 \psi + \frac{1}{2}\left[1 - \frac{(1+A^2)}{(1+\psi)^2}\right] \approx 0 \end{cases} \quad \cdots (1)$$

These equations describe how a wake is initiated by the laser, and how the laser is in turn modified based on the feedback from the wake. Since we made assumptions about the magnitude of various terms, if needed for the physics under study, we can go back and add the relevant terms.

It is possible for us to analytically solve this system of equations in the linear limit, i.e. where the quantities are small, $\psi \ll 1$ & $A^2 \ll 1$

$$\Rightarrow \frac{1}{1+\psi} = 1 - \psi + \psi^2 + \cdots \approx 1 - \psi$$

$$\frac{1}{(1+\psi)^2} = 1 - 2\psi + 3\psi^2 + \cdots \approx 1 - 2\psi$$

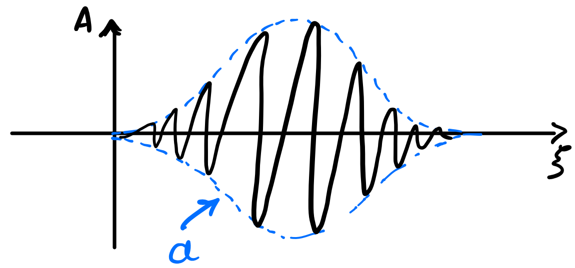
Substituting these terms in eqn 1 gives

$$\begin{cases} 2\,\partial_\xi \partial_s A - \nabla_\perp^2 A + (1-\psi) A = 0 & \cdots (23) \\ \partial_\xi^2 \psi + \psi = \frac{A^2}{2} & \cdots (24) \end{cases}$$

Eqn 23 & 24 are Linear versions (in ψ) of eqn 1.

Similar to the single particle motion lecture, we define the vector potential of the laser in terms of a fast oscillating component multiplied by a slowly varying envelope of the laser:

$$\vec{A} = \frac{a}{2}(\xi, x_\perp, s)\, e^{-ik_0\xi} + c.c.$$



where $\quad \partial_\xi a \ll k_0$

Substituting the equation for the vector potential, we get an equation for be complex envelope of 'a'

$$2\,\partial_\xi \partial_s a - i\,2\omega_0 \partial_s a - \nabla_\perp^2 a + (1+\psi)\,a = 0 \quad \ldots (25)$$

$$\partial_\xi^2 \psi + \psi = \frac{a a^*}{4} + \left(\frac{a^2}{2} e^{-i2\omega_0 \xi} + c.c.\right) \quad \ldots (26)$$

Equation 26 implies that there is a wake response due to the slowly oscillating envelope (first term on the RHS) and a wake response due to the fast oscillating component (second harmonic term on the RHS).

One can show that the wake response to the second harmonic term is much smaller than the wake response to the slowly varying envelope term (HW problem!). Therefore we drop the second term on the RHS to get:

$$\boxed{\partial_\xi^2 \psi + \psi = \frac{|a|^2}{4}} \quad \ldots (27)$$

The $\frac{|a|^2}{4}$ term is called the ponderomotive potential, ϕ_p

Note that there are no derivatives with respect to 's' in equation 27. One way to think about the coupling between equations 25 and 27 is that equation 27 can be applied to the value of the laser profile at some particular 's' to find the wake. The wake then is used in equation 25 to allow you to propagate laser profile in 's'. The essence of quasistatic approximation is that the wake and the laser evolve on completely different scales, which means the wake does not depend on how the laser evolves in 's', but only on the profile of the laser at that particular 's'.

Now, since ψ only weakly depend on 's' & we have already said that we are considering a case where $\nabla_\perp \sim 0$, Eqn 27 becomes

$$\frac{d^2\psi}{d\xi^2} + \psi = \frac{|a|^2}{4} \quad \ldots (28)$$

from eqn 28, ψ can be obtained using Green's function:

$$\frac{d^2 G}{d\xi^2} + G = \delta(\xi - \xi')$$

One can show (HW) that

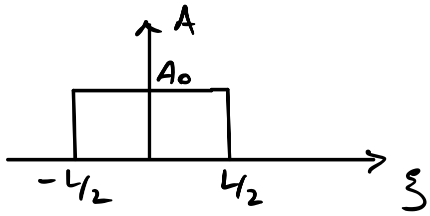
$$G = \eta(\xi - \xi')\,\sin(\xi - \xi'), \quad \cdots (29)$$

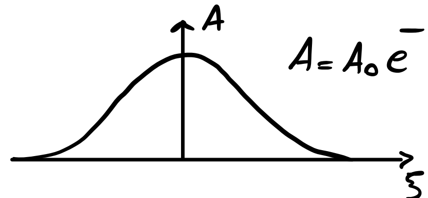
where η is the Heaviside step function & so

$$\psi = \int_{-\infty}^{\infty} d\xi'\, \eta(\xi - \xi')\,\sin(\xi - \xi')\,\phi_p(\xi')$$

$$= \int_{-\infty}^{\xi} d\xi'\,\sin(\xi - \xi')\,\phi_p(\xi') \quad \cdots (30)$$

As an exercise, use a top-hat or Gaussian profile to find the response of the wake; i.e.





$$A = A_0 e^{-(\xi - \xi_0)^2/\tau_0^2}$$

Now that we discussed how lasers generate a wakefield in plasma, we are going to discuss how we can extend this discussion to particle beam drivers in the context of a more generalized form of linear theory. Understanding of a linear theory of beams in plasma is important even for those researching laser wakefield acceleration because regardless of the drive, the beam is going to be an electron beam.

We starting by looking at the laser again, and this time, we start from linear fluid Maxwell equations:

$$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial B_1}{\partial t} \qquad \vec{\nabla} \times \vec{B}_1 = \mu_0 \Big[-e n_0 \vec{v}_{1e} + \overbrace{\vec{J}_b}^{= \rho_b c \hat{z}}\Big] + \frac{1}{c^2}\partial E/\partial t$$

$$\vec{\nabla} \cdot \vec{E}_1 = -e/\epsilon_0 (n_{1e}) + \overbrace{q_b n_b}^{\rho_b}/\epsilon_0 \qquad \vec{\nabla} \cdot \vec{B}_1 = 0 \quad \cdots (31)$$

note that since ions don't move in this high freq. wave, n_{1i} & V_{1i} are both zero.

continuity eqn: $\frac{\partial n_1}{\partial t} + n_0 \vec{\nabla}\cdot(\vec{v_1}) = 0$ ... (32)

For momentum eqn, we use the form of Euler's equation derived after setting vorticity to zero (i.e. eqn 9):

$$\text{eqn } 9 \rightarrow \frac{\partial P}{\partial t} = -eE - mc^2 \nabla \gamma$$

For the Laser, $\gamma = \sqrt{1+P^2} = (1+P_L^2)^{1/2} \sim 1 + \frac{1}{2}P_L^2 = \phi_p$

Note that this equation implicitely assumes $P \ll 1 \Rightarrow \gamma \approx 1$.

Thus γ can be dropped from the term on the LHS also to get

$$m \frac{\partial \vec{v}}{\partial t} = -e\vec{E} - mc^2 \nabla \phi_p \quad \ldots (33)$$

Once again, we are Looking for a wave response, so we take $\frac{\partial}{\partial t}$ of continuity eqn (eqn 32):

$$\frac{\partial^2 n}{\partial t^2} + n_0 \vec{\nabla}\cdot \frac{\partial \vec{v}}{\partial t} = 0$$

$$33 \Rightarrow \quad \frac{\partial^2 n}{\partial t^2} + n_0 \vec{\nabla}\cdot \left[-\frac{eE_1}{m} - c^2 \nabla \phi_p\right]$$

$$\Rightarrow \frac{\partial^2 n}{\partial t^2} - \frac{e n_0}{m} \underbrace{\vec{\nabla}\cdot E_1}_{\downarrow \text{Gauss's Law}} - c^2 n_0 \nabla^2 \phi_p = 0$$

$$\Rightarrow \frac{\partial^2 n}{\partial t^2} - \frac{e n_0}{m \epsilon_0}(-e n_1 + \rho_b) - c^2 n_0 \nabla^2 \phi_p = 0$$

$$\boxed{\Rightarrow (\partial_t^2 + \omega_{p_0}^2)\frac{n_1}{n_0} = c^2 \nabla^2 \phi_p + \omega_{p_0}^2 \frac{\rho_b}{e n_0}} \quad \ldots (34)$$

So the plasma density perturbation is driven by the ponderomotive potential and by the beam. One can immediately see a difference between a laser and the particle beam driver. For the particle beam driver the wakefield is driven by the value of the beam density at that location. The laser however does not have fields that extend outside of it and so the density is driven by the derivative and the gradient of the ponderomotive potential.

Next, we will look for an equation for E_1.

We start as usual by the $\vec{\nabla}\times\vec{\nabla}\times\vec{E}$

$$-\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$ use Maxwell's eqns (Eqn 31)

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{E_1}) + \nabla^2\vec{E_1} = \frac{\partial}{\partial t}\left[-\mu_0 e n_0 \vec{v_1} + \mu_0 \rho_b c(\hat{z}) + \frac{1}{c^2}\frac{\partial \vec{E_1}}{\partial t}\right]$$

Substitute for $\frac{\partial \vec{v_1}}{\partial t}$ from 33 & $\vec{\nabla}\cdot\vec{E}$ from 31:

$$-\frac{1}{c^2}\frac{\partial^2\vec{E_1}}{\partial t^2} + \nabla^2\vec{E_1} - \vec{\nabla}\left(-\frac{e}{\epsilon_0}n_1 + \frac{\rho_b}{\epsilon_0}\right) = -\mu_0\epsilon_0\frac{e n_0}{m\epsilon_0}\left(-eE - mc^2\nabla p\right)$$
$$+\mu_0 c\frac{\partial \rho_b}{\partial t}\hat{z}$$

$$\Rightarrow \boxed{-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \nabla^2\vec{E_1} - \frac{\omega_{p_0}^2}{c^2}\vec{E_1} = -\frac{e}{\epsilon_0}\nabla n_1 + \frac{\nabla\rho_b}{\epsilon_0} + \frac{m}{e}\omega_{p_0}^2\nabla\phi_p + \mu_0 c\frac{\partial\rho_b}{\partial t}\hat{z} \quad \cdots (35)}$$

So equation 34 allows for the calculation of the density perturbation, and the resulting electric field is calculated from equation 35.
This derivation was first done in a 1D analysis by Tom Katsouleas in a paper titled “beam-loading in plasma accelerators”. The next part having to do with the analysis of the wake potential was an insight courtesy of Julian Schwinger when he was a professor at UCLA!

Now, since we are analyzing the accelerating field, we consider the $\hat{z}$ component of equation 35:

$$\Rightarrow -\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} + \nabla^2 E_z - \frac{\omega_{p_0}^2}{c^2}E_z = -\frac{e}{\epsilon_0}\frac{\partial n_1}{\partial z} + \frac{m}{e}\omega_{p_0}^2\frac{\partial\phi_p}{\partial z}$$
$$+\frac{1}{\epsilon_0}\frac{\partial\rho}{\partial z} + \frac{1}{c\epsilon_0}\frac{\partial\rho}{\partial t} \quad \cdots (36)$$

Now, using the co-moving coordinates as above, we make the quasi-static approximation:

$$\partial_z \rightarrow -\partial_\xi$$
$$\partial_t \rightarrow c\,\partial_\xi$$

Recall, for wave operator $\frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2 \rightarrow -\nabla_\perp^2$

meaning the last two terms in eqn 36 cancel out:

$$\frac{1}{\epsilon_0}\frac{\partial \rho}{\partial z} + \frac{1}{\epsilon_0 c}\frac{\partial \rho}{\partial t} = \frac{1}{\epsilon_0}\left[-\frac{\partial \rho}{\partial \xi} + \frac{\cancel{c}}{\cancel{c}}\frac{\partial \rho}{\partial \xi}\right] = 0 \quad \ldots 37$$

Physically, this means that to the first order, the beam charge density does not contribute to the longitudinal electric field for the beam that moves at the speed of light.

$$36, 37 \Rightarrow \left[\nabla_\perp^2 - \frac{\omega_{p0}^2}{c^2}\right] E_z = \frac{e}{\epsilon_0}\partial_\xi n_1 - \frac{m}{e}\omega_{p0}^2\,\partial_\xi \phi_p \quad \ldots 38$$

To get the wake potential, we realize that

$$E_z = \partial_\xi \psi \Rightarrow \psi = \int E_z\, d\xi$$

$$\rightarrow \left[\nabla_\perp^2 - k_p^2\right]\psi = \frac{e n_0}{\epsilon_0}\left[\frac{n_1}{n_0} - \frac{m\epsilon_0}{e^2 n_0}\omega_{p0}^2\,\phi_p\right]$$

$$\Rightarrow \boxed{\left[\nabla_\perp^2 - k_p^2\right]\psi = \frac{e n_0}{\epsilon_0}\left[\frac{n_1}{n_0} - \phi_p\right]} \quad \ldots 39$$

Comparing equations 34 and 39, one can realize that equations 35 can be written in co-moving coordinates in terms of the term on the right hand side of equation 39. So we define

$$\chi = \left[\frac{n_1}{n_0} - \phi_p\right] \quad \ldots 40$$

$$34 \Rightarrow \boxed{\left(\partial_\xi^2 + k_p^2\right)\chi = k_p^2\frac{n_b}{n_0}\frac{q_b}{e} - k_p^2\left[1 - k_p^2\nabla_\perp^2\right]\phi_p} \quad \ldots 41$$

Equations 39 and 41 represent two coupled equations for lasers and particle beams. The beam and the laser create χ . Once we know χ , we can solve for ψ. We can then use ψ to calculate all the forces on the trailing bunch we want to accelerate. We can also use ψ to determine the evolution of drive laser or particle beams.

Let's now go to normalized units and look for a procedure to solve these equations. Recall the normalization for ψ is

$$\frac{e\psi}{mc^2} \rightarrow \psi$$

multiply 39 by $\frac{e}{mc^2} \Rightarrow \left[\nabla_\perp^2 - k_p^2\right] \frac{e\psi}{mc^2} = \left(\underbrace{\frac{e^2 n_0}{m\epsilon_0} \cdot \frac{1}{c^2}}_{k_p^2}\right) \chi$

χ is already normalized

normalize 39 $\rightarrow$
$$\boxed{\left[\nabla_\perp^2 - 1\right]\psi = \chi \quad \cdots (42)}$$

41 $\rightarrow$
$$\boxed{\left[\partial_\xi^2 + 1\right]\chi = q_b n_b - \left[1 - \nabla_\perp^2\right]\phi_p \cdots (43)}$$

Since the properties of the wake are determined by ψ, I am going to combine equations 42 and 43 into a single equation for ψ:

operate on 42 with $\left[\partial_\xi^2 - 1\right]$ & substitute the result in 43 to get

$$\boxed{\left[\partial_\xi^2 + 1\right]\left[\nabla_\perp^2 - 1\right]\psi = \underbrace{q_b n_b}_{\text{beam (drive or witness)}} - \left[1 - \nabla_\perp^2\right]\underbrace{\phi_p}_{\text{Laser}}} \cdots (44)$$

Equation 44 is linear wakefield theory in one equation. As an aside, we can have a witness laser pulse, where we can analyze the photons comprising the laser undergoing acceleration/deceleration depending on the derivative of wake potential. We will come back to this in the "nonlinear optics of plasmas" lecture.

To solve this linear equation, we use Green's function. We also look at the case of the laser and electron beam separately. We also note that since this is a linear equation, if the particle beam and laser are simultaneously present, the total wake will be the linear superposition of the wakes excited by each source term. The physical interpretation is that within the confines of linear theory, each source is going to generate its own wake response & it doesn't really matter that the wake was already perturbed by another source. This fact will give us the framework for the study of beam loading later on.

Laser

In the regions where there is only laser and the beam term drops off, we recover eqn 27 by dropping the $[\nabla_\perp^2 - 1]$ operator from both sides.

$$(\partial_\xi^2 + 1)\,\psi = \phi_p \quad \cdots (45)$$

We use Green's function again, but now, we let the Green's function have transverse gradients to account for the transverse profile of the laser. From Eqn 27/45, it is clear that the transverse gradient of the wake potential will follow that of the laser (side there there is no transverse gradient in this equation). Therefore, if you want the wake potential to have a certain transverse profile (for example to achieve a certain focusing force, $\vec{F}_\perp \sim \vec{\nabla}_\perp \psi$), you will need to have a laser with the same transverse profile.

$$[\partial_\xi^2 + 1]\,G = \delta(\xi - \xi')\,\delta(x_\perp - x_\perp')$$

Apply separation of variables to G_L (L for Laser)

$$G_L = G_{\xi L}(\xi)\;G_{x_\perp L}(x_\perp)$$

$$\Rightarrow G_{x_\perp L}(x_\perp)\,[\partial_\xi^2 + 1]\,G_{\xi L} = \delta(\xi - \xi')\,\delta(x_\perp - x_\perp') \quad \cdots (46)$$

Equation 46 is satisfied if

$$(\partial_\xi^2 + 1)\,G_{\xi L} = \delta(\xi - \xi')$$

$$G_{x_\perp L} = \delta(x_\perp - x_\perp')$$

$$\therefore\; G_L(\xi - \xi', x_\perp - x_\perp') = \eta(\xi - \xi')\sin(\xi - \xi')\,\delta(x_\perp - x_\perp') \quad \cdots (47)$$

$$\Rightarrow \psi = \int_{-\infty}^{\infty} d\xi\; \eta(\xi - \xi')\,\sin(\xi - \xi') \int_{-\infty}^{\infty} dx_\perp\; \delta(x_\perp - x_\perp')\;\phi_p(\xi, x_\perp)$$

If the ponderomotive potential seperable (as is the case for a regular laser pulse):

$$\phi_p = \phi_{p\xi}(\xi')\;\phi_{px_\perp}(x_\perp') \quad \cdots (48)$$

$$\Rightarrow \boxed{\psi = \phi_{p_\perp}(x_\perp) \int_{-\infty}^{\xi} d\xi'\; \sin(\xi - \xi')\;\phi_p(\xi, x_\perp)} \quad \cdots (49)$$

Note that the transverse component of ψ is directly dictated by the transverse profile of the ponderomotive potential as expected.

<u>Particle Beam</u>

The equations in this case are different, but the same logic will apply

$$(\partial_\xi^2+1)[\nabla_\perp^2-1]\,\psi = \underbrace{q\,n_b}_{=\rho_b} \qquad (\phi_p=0) \quad \cdots (50)$$

We apply the separation of variables to G again:

$$G_b = G_{\xi b}(\xi)\, G_{x_\perp b}(\vec{x}_\perp)$$

$$\Rightarrow \quad (\partial_\xi^2+1)(\nabla_\perp^2-1)\, G_{\xi b}\, G_{x\perp b} = \delta(\xi-\xi')\,\delta(x_\perp - x'_\perp)$$

Thus we obtain two equations for the Green's fcn:

$$(\partial_\xi^2+1)\, G_{\xi b} = \delta(\xi-\xi') \quad \cdots (51)$$

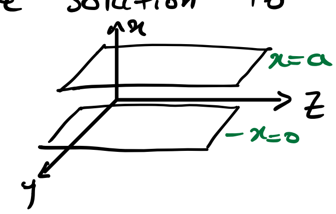
$$(\nabla_\perp^2-1)\, G_{x_\perp b} = \delta(\vec{x}_\perp - x'_\perp) \quad \cdots (52)$$

$$51 \Rightarrow G_{\xi b} = \eta(\xi-\xi')\sin(\xi-\xi') \cdots (53) \quad \text{(Same as Laser)}$$

$$G_{\perp b} = -\frac{K_0(r-r')}{2\pi} \quad \cdots (54)$$

where K_0 is the modified Bessel fcn.

Eqn 54 is the answer for a cylindrically symmetric geometry. As a HW try finding the solution to Eqn 52 in a 2D slab geometry, i.e.



Note that while K_0 goes to infinity like the delta function, the physical response is obtained through the integration of the Green's function. So as long as the integration gives a physically meaningful response, the behavior of Green's function itself is of little consequence.

For the wake function then, we get

$$\psi = \int_{-\infty}^{\infty} d\xi' \, \eta(\xi-\xi') \sin(\xi-\xi') \int_0^{2\pi} \frac{d\phi'}{2\pi} \int_0^{\infty} dr' \, r' \left(-K_0(r-r')\right) \rho_b(r';\xi) \quad \ldots (55)$$

Now if the charge density is some complicated function with correlations, etc, evaluating Eqn 55 is going to be nontrivial. For practical purposes, we are often interested in the peak accelerating field, which occurs near the axis. In this case, and assuming that beam charge density is separable,

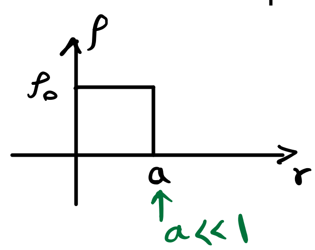
$$\rho_b = \rho_{\xi b}(\xi') \, \rho_{x_\perp b}(r') \quad \ldots (56)$$

$$\psi(\xi, r=0) = \int_{-\infty}^{\infty} d\xi' \, \eta(\xi-\xi') \sin(\xi-\xi') \, \rho_{\xi b}(\xi') \underbrace{\int_0^{\infty} dr' \, r' \, K_0(r') \, \rho_{x_\perp b}(r')}_{\equiv R(0) \text{ radial function evaluated at the origin.}} \quad \ldots (56)$$

The function R(0) is what distinguishes a beam response from a laser response. Consider the following cases for a beam that's small compared to a skin depth:

(a) a flat top profile:

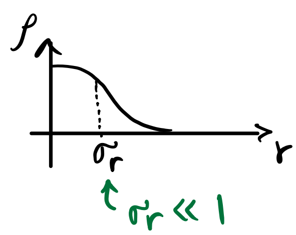
$$R(0) = \frac{a^2}{2}\left[0.06159 - \ln(a)\right]$$



(b) a Gaussian profile:

$$\rho = \exp\left(-r^2/2\sigma_r^2\right)$$

$$R(0) = \sigma_r^2\left[0.05797 - \ln(\sigma_r)\right]$$



It can be observed that the answer is relatively insensitive to the shape of the beam. Moreover, the answer is relatively insensitive to beam width & as we already know from the single particle lecture, the fields of the drive beam scale as charge per unit length. For the Gaussian beam for example

$$\left.\begin{array}{l} \psi \sim \rho_{b0} R_0 \sim \rho_{b0} \, \sigma_r^2 \ln\left(\frac{1}{k_p \sigma_r}\right) \\ \rho_{b0} = \frac{N}{(2\pi)^{3/2} \sigma_r^2 \sigma_z} \end{array}\right\} \Rightarrow \psi \sim \underset{\text{charge per unit Length}}{\frac{N}{\sigma_z}} \; \underset{\text{slow function of } \sigma_r}{\ln\left(\frac{1}{k_p \sigma_r}\right)}$$

So if you take a beam with fixed amount of charge and start making it narrower and narrower, the wake is very insensitive to how narrow you make it. This was not appreciated at the time of the beam loading paper by Katsouleas.

If we do the full analysis and ask the question of how far beyond the beam the effect of R(0) is felt, you will see that the beam wake extends roughly to a skin depth, even when the beam is much narrower. Therefore when the beam is absorbing wake's energy, it can do so out to a skin depth. This latter fact distinguishes the beam case from the laser, for which the wake exist only where the laser is.

To analyze the first term, we consider again the case of a Gaussian profile

$$\int_{-\infty}^{\infty} d\xi' \, \eta(\xi-\xi') \sin(\xi-\xi') \, e^{-\xi^2/2\sigma_z^2}$$

$$= \int_{-\infty}^{\xi} d\xi' \sin(\xi-\xi') \, e^{-\xi'^2/2\sigma_z^2}$$

for $\xi \gg \sigma_z$ (i.e. looking way behind the bunch)

$$\simeq \int_{-\infty}^{\infty} d\xi' \sin(\xi-\xi') \, e^{-\xi'^2/2\sigma_z^2}$$

With these new limits, this integral can be evaluated analytically (HW). The electric field can then be calculated as

$$E_z = \frac{\partial \psi}{\partial \xi}$$

$$E_z(\xi, r=0) = \sqrt{2\pi}\, \sigma_z \, e^{-\sigma_z^2/2} \cos\xi \; R(0) \; \rho_{b0} \ldots (57)$$

unnormalized

$$\Rightarrow \frac{eE_z}{mc\omega_p} = \sqrt{2\pi}\, \cancel{k_p\sigma_z}\, e^{-k_p^2\sigma_z^2/2} \cos k_p\xi \; \frac{N}{\cancel{k_p\sigma_z}\, n_0} \; \ln\left(\frac{1}{k_p\sigma_r}\right)$$

What is the maximum of E?

$$\frac{eE_z}{mc\omega_p} = \sqrt{2\pi}\, e^{-k_p^2\sigma_z^2/2} \cdot \frac{N}{n_0} \ln\left(\frac{1}{k_p\sigma_r}\right)$$

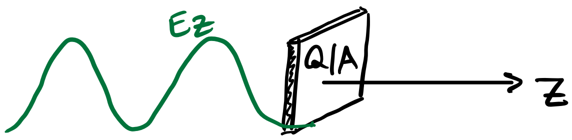
Check this!

Transformer ratio: transformer ratio is a measure of efficiency of an accelerator. It is the peak accelerating field divided by the peak decelerating field. The higher the transformer ratio, the faster the energy transfers from the drive to trailing bunch. Consider a short ($\sigma_z \ll c/\omega_p$) and wide ($\sigma_r \gg c/\omega_p$) drive bunch. This profile is essentially a delta function in ξ. In this case, the beam density is represented by a surface charge density, $\sigma = Q/\text{Area}$.

Note that for such a function, a laser and particle driver have the same effect since the transverse gradients from equation 50 drop out making it the same as equation 45, just with the particle beam drive term.

$$\Rightarrow \quad \varphi = -\eta(\xi-\xi')\sin(\xi-\xi')\,\sigma$$

$$E_z = -\eta(\xi-\xi')\cos(\xi-\xi')\,\sigma$$



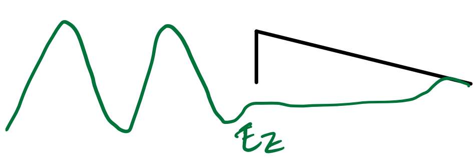
Physically, the delta function excites a wake starting from the position of the beam. Because of the discontinuity of the wake at the location of the beam, the decelerating force felt by the wake is

$$E_{w-b} = \frac{E_{front} + E_{back}}{2} = \frac{E_z}{2}$$

On the other hand, the accelerating field is the full field of the wake, E_z

$$\therefore \text{ ratio} = R = \frac{E_z}{E_{w-b}} = 2$$

In fact, there is a fundamental theorem that states that for a symmetric beam the transformer ratio is limited to less than two (see K. Bane and A. Chao 1985). Therefore to achieve a transformer ratio higher than 2, one has to use an asymmetric bunch:



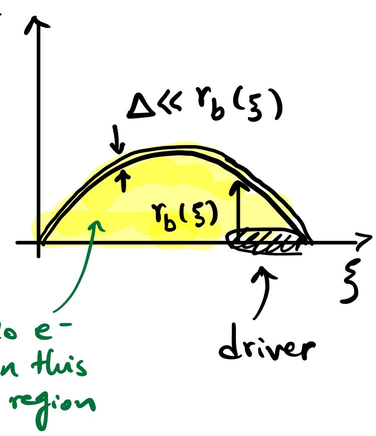
Nonlinear plasma wake excitation

In general the case of nonlinear wake excitation is quite complex and simple analytical formulas are not as extensively developed. In the case of the blowout regime, the properties of the wake function are related to the trajectory of the innermost electron.

Recall,

$$\begin{cases} \psi(\xi, r) = \psi_0(\xi) - \dfrac{r^2}{4} \\ \psi_0(\xi) = (1+\beta)\dfrac{r_b^2}{4} \approx \dfrac{r_b^2}{4} \quad (\text{for } r_b \gg 1) \end{cases}$$

Eqn 18 & 20 in the "3D blowout wakefields" notes



From the transverse component of the equation of motion for the innermost e^-, in the case of $r_b \gg 1$, we got (Eqn 39 in the "3D blowout wakefields" notes)

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = 0$$

The addition of a driver (in particular a particle beam driver) modifies the source terms inside the wakefield in the following way

$$\rho = \rho_0 - en_b = 1 - n_b \quad \text{in normalized units}$$

$$J_z = 0 - en_b c = -n_b$$

$$\Rightarrow \begin{cases} \phi = \phi_0(\xi) - \dfrac{r^2}{4} + \lambda(\xi) \ln r & \dots (58) \\ A_z = A_{z0}(\xi) + \lambda(\xi) \ln r & \dots (59) \end{cases}$$

$$A_r = -\frac{r}{2} \frac{d\psi_0}{d\xi} \quad \text{as before}$$

Here, $\lambda(\xi) = \int_0^{r > r_b} -n_b(r', \xi) 2\pi r' dr'$, where a thightly focused beam is assumed such that we are mostly interested in fields outside of the beam $r > r_b$

Following the same procedure as the "3D blowout wakefield" notes, one can show that eqn 3a now turns to

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi^2}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2} \quad \ldots (60)$$

Given a profile for $\lambda(\xi)$, one can in theory calculate $r_b(\xi)$ via numerical integration with initial conditions $r_b|_{\xi_i} = 0$ & $\frac{dr_b}{d\xi}\Big|_{\xi_i} = 0$. Once r_b is found, one can determine φ & therefore the properties of the wakefield using the equations on previous pg. However, in a paper in 2021, Golovonov, et al. realized that at $r_b \to 0$, the RHS $\to \infty$, forcing the $\frac{d^2 r_b}{d\xi^2}$ to diverge. Their solution consisted of a power law substitution,

$$\Psi_\xi(\xi) = \frac{r_b^2(\xi)}{4} \quad \ldots (61)$$

As it turns out the parabolic substitution is the only one elliminating divergence of second derivative w/o introducing zero solution into the eqn. Substituting eqn 61 into 60, we get

$$\frac{d^2\Psi_\xi}{d\xi^2} + \frac{1}{2\Psi_\xi}\left(\frac{d\Psi_\xi}{d\xi}\right)^2 = \frac{\lambda}{2\Psi_\xi} - \frac{1}{2} \quad \ldots (61)$$

For this new set of variables, the second derivative is finite

& $\frac{1}{\Psi_\xi}\left(\frac{d\Psi_\xi}{d\xi}\right)^2$ helps cancel the 0/0 singularity.

One thing to note is that the LHS is positive definite &

so this eqn is valid for $\left.\frac{\lambda}{2\Psi_\xi}\right|_{r_b \to 0} \approx |\rho_b(\xi,0)| > \frac{1}{2}$

In this case, where $r_b < \sigma_b$, we make the approximation that the eqn for blowout radius still applies, except that instead of $\lambda(\xi)$, we use $\lambda(\xi,r) = \int_0^r \rho(r';\xi)\, 2\pi r' dr'$

inital condition then are:

$r_b(0)=0 \Rightarrow \Psi(\xi_0)=0$

$\frac{dr_b}{d\xi}=0 \Rightarrow \left.\frac{1}{\Psi_\xi}\left(\frac{d\Psi_\xi}{d\xi}\right)^2\right|_{\xi=0} = 0$

Finally, comparing the definition of Ψ_ξ & Ψ we realize

$$\Psi(\xi,r) = \Psi_\xi - \frac{r^2}{4}$$

$$E_z = \frac{\partial\Psi}{\partial\xi} = \frac{d\Psi_\xi}{d\xi}$$