Studies of Chiral Gauge Theories and Heavy-Quark Hadrons

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Abstract of the Dissertation

Studies of Chiral Gauge Theories and Heavy-Quark Hadrons

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In this dissertation we study two aspects of strongly coupled quantum field theories: ultraviolet to infrared evolution and non-perturbative behavior of chiral gauge theories and properties of hadronic bound-states in quantum chromodynamics (QCD). We first calculate the ultraviolet to infrared evolution and analyze possible types of infrared behavior for several families of asymptotically free chiral gauge theories with gauge group $\mathrm{SU}(N)$ and massless chiral fermions transforming according to higher-dimensional representations.

Next, we analyze patterns of dynamical gauge symmetry breaking in strongly coupled chiral gauge theories with direct-product gauge groups. We find that the symmetry-breaking behavior depends sensitively on the relative sizes of the gauge couplings of the different factor groups in the direct product.

Finally, we study the radiative decay of a 1^{+-} heavy $Q\bar{Q}$ meson via the channel $1^{+-} \to 0^{-+} + \gamma$ in the covariant light-front quark framework. In particular, we calculate the decay widths for the specific decay channels $h_c(1P) \to \eta_c(1S) + \gamma$ and $h_b(1P) \to \eta_b(1S) + \gamma$. We compare our results with experimental data, finding reasonable agreement.

Dedication Page

To my parents

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- H.-W. Ke, X.-Q. Li, Y.-L. Shi, G.-L. Wang and X.-H. Yuan, "Is $Z_b(10610)$ a Molecular State?", JHEP **1204**, 056 (2012) [arXiv:1202.2178] (16 pages).
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Chapter 1

Introduction

In this chapter, we will briefly review some basic concepts of chiral gauge theories and discuss the motivation for studies of chiral gauge theories. We also give a brief introduction to the radiative decay of heavy-quark hadrons. We present an outline of this thesis at the end of this chapter. The content of this thesis is mainly based on published papers [1–5].

1.1 Chiral Gauge Theories

1.1.1 Chiral Gauge Symmetry

In quantum field theories that contain gauge fields and massless matter field (fermions), we have the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{m} \bar{\psi}_{a,i,L} i \not{D}_L \, \psi^a_{i,L} + \sum_{j=1}^{n} \bar{\chi}_{b,j,R} i \not{D}_R \, \chi^b_{j,R} \ , \tag{1.1}$$

where a, b and i, j denote gauge and flavor indices, respectively. We assume left-handed fermions ψ couple to gauge group G_L in R_1 representation and right-handed fermions χ couple to gauge group G_R in R_2 representation. For most of our studies, $G_L = G_R$, but here we keep a formal distinction between these groups. The covariant derivatives are defined as $\not{D}_L = \gamma^\mu(\partial_\mu - i g_L A^\alpha_{L,\mu} T_{L,\alpha})$ and $\not{D}_R = \gamma^\mu(\partial_\mu - i g_R A^\alpha_{R,\mu} T_{R,\alpha})$, where $g_L,\ g_R$ denote the gauge field couplings, $A^\alpha_{L,\mu},\ A^\alpha_{R,\mu}$ denote gauge fields and $T_{L,\alpha},\ T_{R,\alpha}$ are generators of the Lie algebras of these groups, acting in the space of the given representations of matter fields. The indices L, R represent the properties that the gauge groups act on left- and right-handed fermions, respectively. Here for the purpose of illustration, we only define gauge groups to be G_L and G_R. One can easily generalize the discussion here to more complicated situations. The index α is the index of generators of groups. $\psi^{a,}_{i,L}$ represents the left-handed fermionic fields

$$\psi^a_{i,L} = \left(\frac{1-\gamma^5}{2}\right)\psi^a_i \ , \tag{1.2}$$

and $\chi^{a,}_{i,R}$ represents the right-handed fermionic fields

$$\chi^b_{j,R} = \left(\frac{1+\gamma^5}{2}\right)\chi^b_j \ . \tag{1.3}$$

Under a gauge transformation, ψ_L transforms as

$$\psi_L \to U_1\psi_L \ , \quad U_1 \in G_L(R_1) \ , \tag{1.4}$$

and

$$\chi_R \to U_2\chi_R \ , \quad U_2 \in G_R(R_2) \ . \tag{1.5}$$

With no loss of generality, we can rewrite all fields in terms of left-handed spinors. Then the kinetic term in the lagrangian can be rewritten in terms of left-handed spinors

$$\sum_{j=1}^{n} \bar{\chi}_{b,j,R} i \not{D}_R \, \chi^b_{j,R} \to \sum_{j=1}^{n} \bar{\chi}'_{b,j,L} i \not{D}'_R \, \chi'^b_{j,L} \tag{1.6}$$

where the renamed left-handed fermion field χ'_L transforms in the conjugate representation $\bar{R}_2$ of G_R:

$$\chi'_L \to U_2^*\chi'_L \ , \quad U_2^* \in G_R(\bar{R}_2) \ . \tag{1.7}$$

In the special case where $G_L = G_R$, the numbers of left-handed fermions ψ_L and right-handed fermions $\chi_R(\chi'_L)$ are the same, and the representations satisfies condition $R_1 = R_2$, then the fermions are in a real representation of the gauge group. We call such theories vectorial gauge theories. Quantum chromodynamics (QCD) in the Standard Model is an example of a vectorial gauge theory. Another special case is where R_1 and R_2 are both real or pseudo-real, leading again to a vectorial gauge theory.

Apart from these special cases, the gauge field theory is intrinsically chiral, so left- and right-handed fermions transform differently under gauge transformation. We call such quantum field theory as chiral gauge theory. We define a chiral gauge theory as irreducibly chiral if it does not contain any vectorial subset among its fermions.

1.1.2 Global chiral symmetry

In the absence of Dirac and Majorana mass terms,

$$M_D^{ij}\bar{\psi}_{a,i,L}\chi^a_{j,R} + M_L^{ij}(\psi^a_{j,R})^T C\psi^a_{i,R} + M_R^{ij}(\chi^a_{j,R})^T C\chi^a_{j,R} + h.c. \;, \tag{1.8}$$

the theory is invariant under the following global flavor symmetry transformation:

$$\psi^a_{i,L} \to (U_L)^k_i \psi^a_{k,L} \;, \quad (U_L)^i_j \in U(m) = SU(m) \otimes U_1(1) \;, \tag{1.9}$$

$$\chi^a_{j,R} \to (U_R)^k_j \chi^a_{k,R} \;, \quad (U_R)^i_j \in U(n) = SU(n) \otimes U_2(1) \;. \tag{1.10}$$

Hence the theory has a classical global flavor symmetry:

$$G_f = SU(m) \otimes SU(n) \otimes U_1(1) \otimes U_2(1) \;. \tag{1.11}$$

Due to instantion effects of gauge interaction [6], classical global U(1) symmetries $U_1(1)$ and $U_2(1)$ are broken. Depending on the content of gauge groups, one may find a residual unbroken global $U(1)$ symmetry which is a linear combination of $U_1(1)$ and $U_2(1)$. Detailed discussions on the global $U(1)$ symmetries can be found in the following chapters.

1.1.3 Chiral Symmetry Breaking

In an asymptotically free chiral gauge theory, in the renormalization-group (RG) evolution of the theory from the ultraviolet (UV) at high Euclidean energy/momentum scales μ to the infrared (IR) at low μ, the gauge interaction may become strong enough that it produces bilinear fermion condensate(s), of the form

$$\langle\bar{\psi}_{a,i,L}\chi^a_{j,R}\rangle + h.c. \;, \quad \langle(\psi^a_{j,R})^T C\psi^a_{i,R}\rangle + h.c. \;, \quad \langle(\chi^a_{j,R})^T C\chi^a_{j,R}\rangle + h.c. \;. \tag{1.12}$$

In an irreducibly chiral gauge theory, these condensates dynamically break the gauge symmetry. They also generically break chiral flavor symmetries.

As a result of the dynamical breaking of the gauge symmetries, the gauge bosons that correspond to broken generators of the gauge group become massive. For spontaneously broken global flavor chiral symmetries, there will also be massless Nambu-Goldstone bosons (NGBs).

1.1.4 Examples of Chiral Gauge Theories

Chiral gauge theories are an important subject in the study of general quantum field theories. The electroweak part of the Standard Model is an example of a chiral gauge theory in which fermions couple to gauge field in a parity-violating manner. In the Standard Model, left-handed fermions (quarks and leptons)

$$E_{L,i} = \left(\begin{array}{c} \nu_{\ell_{L,i}} \\ \ell^-_{L,i} \end{array} \right) = (\mathbf{2}, -1) \ , \quad Q_{L,i} = \left(\begin{array}{c} u_{L,i} \\ d_{L,i} \end{array} \right) = (\mathbf{2}, 1/3) \tag{1.13}$$

transform in the fundamental representation of weak $SU(2)$ interaction with hypercharges $Y = -1$ and $+1/3$, respectively. While right-handed fermions

$$\ell_{R,i} = (\mathbf{1}, -2) \ , \quad u_{R,i} = (\mathbf{1}, +4/3) \ , \quad d_{R,i} = (\mathbf{1}, -2/3) \tag{1.14}$$

are singlets of $SU(2)$ with hypercharges $Y = -2$, $+4/3$ and $-2/3$.

Another example of chiral gauge theory is the Georgi-Glashow model, which is a particular grand unification theory (GUT) [7]. In this theory, the gauge group is $SU(5)$. Non-singlet left-handed fermions of each generation transform as a $\bar{\mathbf{5}}$ and $\mathbf{10}$ representation of the gauge group.

1.2 Motivation for Studying Chiral Gauge Theories

Chiral gauge theories (without scalars) that are asymptotically free and can therefore become strongly coupled at low energies are of intrinsic field-theoretic interest in their own right. They have been of interest in the past for several specific reasons.

1.2.1 Light Composite Fermions

One motivation involved an effort to understand the pattern of masses of quarks and leptons in the Standard Model. Since the respective lower bounds on the compositeness scales of these Standard-Model fermions are much larger than their masses, a plausible approach was to begin by using a theoretical framework in which they were massless. Strongly coupled irreducibly chiral gauge theories are a natural candidate for such a framework, since the chiral gauge invariance forbids any mass terms. If such a theory satisfies the 't Hooft global anomaly-matching conditions, then, as the gauge coupling becomes sufficiently strong in the infrared, the gauge interaction could confine and produce massless gauge-singlet composite spin-1/2 fermions [8–14]. Additional ingredients were assumed to produce the observed masses of the quarks and leptons. This was a very ambitious program, and although it did not produce an actual calculation of the observed fermion mass spectrum, it introduced a number of intriguing ideas concerning possible compositeness of these fermions that might be part of a deeper theory going beyond the Standard Model.

1.2.2 Dynamical Electroweak Symmetry Breaking and Fermion Mass Generation

A different motivation for studying strongly coupled chiral gauge theories arose in the context of models that sought to explain both dynamical electroweak symmetry breaking and fermion mass generation. In models of dynamical electroweak symmetry breaking [15], a vectorial, asymptotically free gauge interaction would become strongly coupled at the TeV scale and would produce bilinear fermion condensates involving a set of fermions that are nonsinglets under the electroweak gauge group $G_{EW} = SU(2)_L \otimes U(1)_Y$. In this scenario, the interaction that becomes strong is vectorial and breaks the weakly coupled chiral gauge interaction G_{EW} to a vectorial subgroup gauge symmetry, namely, $U(1)_{\rm em}$ of the electromagnetic interaction. Modern theories of this type feature a slowly running coupling due to an approximate IR fixed point of the renormalization group. In turn, this produces an approximately scale-invariant behavior over a substantial interval of Euclidean energy/momenta. When this approximate scale (i.e., dilatation) invariance is broken spontaneously by the formation of the bilinear fermion condensates, a light, approximate Nambu-Goldstone boson, the dilaton, is produced. This dilaton has properties similar to those of the Standard-Model Higgs boson [16]. Thus, in this type of UV extension of the Standard Model, the Higgs boson is composite, and the naturalness problem is solved because its mass is protected by the fact that it is a nearly Nambu-Goldstone boson of an approximate dilatation symmetry in the strongly coupled gauge theory.

In order to give masses to Standard-Model fermions, theories with dynamical electroweak symmetry breaking are embedded in larger theories [17]. These UV extensions are designed to try to explain also the generational hierarchy observed in these fermion masses. A basic property of a chiral gauge theory is that if it becomes strongly coupled, it can produce bilinear fermion condensates that self-break the gauge symmetry [18, 19]. Reasonably UV-complete models for dynamical electroweak symmetry breaking and Standard-Model fermion mass generation made use of this feature (e.g., [1, 20–26]). These involved strongly coupled chiral gauge interactions that led to the formation of various fermion condensates which broke the initial chiral gauge symmetry in a sequence of stages that might plausibly explain the Standard Model quark and charged lepton masses masses and their generational hierarchy. A low-scale seesaw mechanism was presented in [20] that also can provide a dynamical explanation for neutrino masses. This sequential breaking was such as to yield, as a residual symmetry, the vectorial gauge symmetry that is strongly coupled at the TeV scale. Ref. [20] used a direct-product chiral gauge group with two strongly coupled gauge interactions and pointed out that different patterns of sequential gauge symmetry breaking (denoted G_a and G_b in [20]) could occur, depending on the relative sizes of gauge couplings corresponding to these two factor groups. A similar phenomenon was noted in other models studied in [21]. This property of the nonperturbative behavior of direct-product chiral gauge theories will be discussed in detail in Chapter 7. The current data on the 125 GeV Higgs boson discovered at the CERN Large Hadron Collider (LHC) in 2012 are consistent with the predictions of the Standard Model, in which the Higgs is a pointlike particle. However, the question of whether, in fact, the Higgs boson is a pointlike particle or is composite, remains the subject of continuing experimental and theoretical investigation. Theories of dynamical electroweak symmetry breaking that produce a composite Higgs boson as a quasi-dilaton thus remain of interest, both as regards possible Higgs compositeness and because they provide a possible ultraviolet completion of the Standard Model that can explain the origin of the masses and generational hierarchy of quarks and leptons. The role of self-breaking of strongly coupled chiral gauge theories in producing this generational hierarchy is a central aspect of these theories, and further motivates their analysis.

1.2.3 Pattern of Gauge Symmetry Breaking

Another motivation for the present study is the fact that patterns of gauge symmetry breaking by Higgs fields depend on parameters in the Higgs potential V, which one can choose at will, subject to the constraint that V should be bounded from below. This reduces the predictiveness of such theories. In contrast, once one has specified the gauge and fermion content of a chiral gauge theory, together with the values of the gauge couplings at a reference point (which is naturally chosen to be in the deep UV for theories with asymptotically free non-Abelian gauge interactions), then the dynamics determines the pattern of gauge symmetry breaking uniquely [27]. In principle, such theories are thus more predictive than theories with various parameters that can be chosen arbitrarily. For example, in the Standard Model, the coefficient of the quadratic term in the Higgs potential, $\phi^\dagger\phi$, is chosen to be negative, when, in principle it could have been chosen to be positive. Because of this, the Standard Model accommodates, but does not explain, electroweak symmetry breaking, because it gives no fundamental explanation for why the coefficient of the $\phi^\dagger\phi$ term in the Higgs potential was chosen to be negative. This is the analogue of how the σ model describes spontaneous chiral symmetry breaking (SχSB) in QCD; again, it sets the coefficient of a quadratic term in the scalar potential negative in order to induce a vacuum expectation value for the σ field. But the underlying explanation for SχSB in QCD is the dynamical formation of a bilinear quark condensate. The negative coefficient of the quadratic term in the σ model is only a phenomenological device to model this phenomenon. A similar comment applies for the way in which the Standard Model accommodates, but does not really explain, quark and lepton masses and mixing by means of Yukawa couplings. In contrast, chiral gauge theories with sequential symmetry breaking can dynamically produce and hence have the potential to explain, the generational hierarchy observed among the quark and lepton masses.

1.3 Radiative Decays of Heavy Hadrons

We next discuss a second main area of research contained in our Ph.D. thesis. Heavy-quark $Q\bar{Q}$ bound states play a valuable role in elucidating the properties of quantum chromodynamics (QCD). Here, the term "heavy quark" refers to a quark with a mass large compared with the QCD scale, $\Lambda_{QCD} \simeq 0.3$

GeV. Since the discoveries of the J/ψ in 1974 [28,29] and other $c\bar{c}$ charmonium states, and the Υ in 1977 [30,31] and other $b\bar{b}$ states, we now have a very substantial set of data on the properties and decays of these heavy quarkonium states. Some reviews include [32]- [42]. The goal of understanding these data motivates theoretical studies, in particular, studies of the decays of $Q\bar{Q}$ states. Indeed, a number of exotic hadrons containing heavy quarks have also been observed. The present writer coauthored a paper on one of these exotic hadrons (H.-W. Ke, X.-Q. Li, Y.-L. Shi, G.-L. Wang and X.-H. Yuan, Is Zb(10610) a Molecular State?, JHEP 1204, 056 (2012) [arXiv:1202.2178]) as an undergraduate.

Among various decay channels, radiative decays are a very good testing ground for models, since the emitted photon is directly detected, and the electromagnetic interaction is well understood. An electric dipole (E1) transition is one the simplest types of radiative decays. Here we consider E1 transitions of the form

$$^1P_1 \to {}^1S_0 + \gamma, \tag{1.15}$$

where a spin-singlet P-wave $Q\bar{Q}$ quarkonium state decays to a spin-singlet S-wave $Q\bar{Q}$ state. In terms of the spin J and the charge and parity quantum numbers P and C, indicated as J^{PC}, a radiative decay of this type has the form $1^{+-} \to 0^{-+} + \gamma$.

Several theoretical analyses of the E1 transition rates for these decays have been carried out, using various models [43]- [53]. A number of these models utilize the non-relativistic quantum mechanics formula for an E1 transition, involving the calculation of the overlap integral of the quarkonium wavefunctions of the initial and final states. The quarkonium wavefunction is obtained from the solution of the Schrödinger equation with non-relativistic potentials, such as the Cornell potential, $V = -(4/3)\alpha_s(m_Q)/r + \sigma r$. The first term in this potential is a non-Abelian Coulomb potential representing one-gluon exchange at short distances, where $\alpha_s(m_Q) = g_s(\mu)^2/(4\pi)$ is the strong coupling evaluated at the scale of the heavy quark mass, m_Q, and the second term is the linear confining potential, where σ is the string tension. Current data yield a fit to $\alpha_s(\mu)$ such that $\alpha_s \simeq 0.33$ at the scale $\mu = 1.5$ GeV relevant for $c\bar{c}$ states and $\alpha_s \simeq 0.21$ at the scale $\mu = 4.7$ GeV relevant for $b\bar{b}$ states [54]. Relativistic corrections have also been calculated by replacing the Schrödinger equation by the Dirac equation, and computing corrections in powers of v/c, where v is the velocity of the heavy (anti)quark in the rest frame of the $Q\bar{Q}$ bound state.

It is of interest to study the radiative decays (1.15) with a fully relativistic approach, namely the light-front quark model (LFQM) [55]- [65]. This approach naturally includes relativistic effects of quark spins and the internal motion of the constituent quarks. Another advantage of the light-front quark model is that it is manifestly covariant. Hence it is easy to boost a hadron bound state from one inertial Lorentz frame to another one when the bound state wavefunction is known in a particular frame [59]. The light-front approach has been used to study semileptonic and nonleptonic decays of heavy-flavor D and B mesons and also to evaluate radiative decay rates of heavy mesons [66–70].

1.4 Outline of Thesis

This thesis is organized as follows. In Chapter 2 we introduce some background knowledge that is relevant for the UV to IR evolution of chiral gauge theories. In Chapter 3 we discuss our general theoretical framework and methods of analysis. In Chapter 4,5, 6 we study several classes of chiral gauge theories in which fermions are in various representations of a single $SU(N)$ gauge group. In Chapter 7 we study a variety of different chiral gauge theories with a direct-product gauge groups and fermion contents. These involve both unitary and orthogonal gauge groups. Finally, in Chapter 8 we discuss radiative decays of heavy-quark hadrons in the light-front approach.

Chapter 2

Generalities on UV to IR Flows in Chiral Gauge Theories

In this chapter we summarize generalities on UV to IR flows in chiral gauge theories.

2.1 Renormalization-Group

The properties of asymptotically free chiral gauge theories change as functions of the Euclidean momentum scale μ at which one measures these properties. We use the renormalization-group to study momentum scale μ dependence of physical quantities. The Callan-Symanzik equation [71, 72] is one specific form of renormalization-group equations. This has the form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma\right] G^{(n)}(\{x_i\};\mu,g) = 0 \tag{2.1}$$

where $G^{(n)}(\{x_i\};\mu,g)$ is a renormalized n-point correlation function, and γ is an anomalous dimension. The beta function $\beta(g)$ describes the dependence of g on the momentum scale μ:

$$\beta(g) = \mu\frac{d}{d\mu}g = \frac{d}{d\ln\mu}g \ = \frac{dg}{dt} \ , \tag{2.2}$$

where we define $dt = d\ln\mu$. Equivalently, we may define

$$\beta_\alpha = \frac{d\alpha}{dt} \ , \tag{2.3}$$

where $\alpha = g^2/(4\pi)$, so that $\beta_\alpha = [g/(2\pi)]\beta(g)$. Let us define $a = g^2/(16\pi^2) = \alpha/(4\pi)$. The beta function can be expressed as a series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell \tag{2.4}$$

where b_ℓ is the ℓ-th loop coefficient. We are interested in non-Abelian chiral gauge theories which are asymptotically free, i.e. (with the minus sign extracted as above), theories in which the first-loop coefficient $b_1 > 0$. The coupling constant g approaches zero in the UV limit $\mu \to \infty$ and hence these are well-defined free theories in this UV limit. For sufficiently large μ in the deep UV, a theory of this type is weakly coupled and can be described by perturbative methods. As μ decreases, the gauge coupling increases and there can be several infrared (IR) phases of chiral gauge theories.

2.2 Phases of Chiral Gauge Theory

We next discuss the possible types of behavior that might occur as the theory evolves ("flows") from the deep UV to the IR.

2.2.1 Non-Abelian Coulomb Phase

If $b_1 > 0$ and $b_2 < 0$, we can find a zero of the beta function at the two-loop level, which occurs at $\alpha_{IR,2\ell} = -4\pi b_1/b_2$. One can then calculate higher-order corrections to this. Let us denote the IR zero of the beta function as α_{IR}. If α_{IR} is smaller than a critical value α_{cr} that triggers the formation of bilinear fermion condensates, then it is an exact stable IR fixed point (IRFP) of the renormalization group. In this case, the chiral gauge theory will flow to this point in the UV to IR evolution. Here and below we will consider a formal continuation of the number(s) of fermions in various representations from positive integer to positive real values. If the fermion content is such that b_1 is positive but small, and $b_2 < 0$, then this IRFP occurs at a weak value and the IR theory is expected to be in a deconfined non-Abelian Coulomb phase (NACP). This is the analogue, for chiral gauge theories, to a weakly coupled IRFP in vectorial gauge theories discussed in [73].

2.2.2 Confinement without Spontaneous Chiral Symmetry Breaking

In contrast, depending on the gauge group and the fermion content, the theory may become strongly coupled during the UV to IR evolution. For example, the beta function may not have any (perturbative) IRFP, so that the gauge coupling will continue to grow as it flows from the UV to the IR. In the strong-interaction region, if the theory satisfies the 't Hooft global anomaly-matching conditions (See discussion in 3.3), then it might confine and produce massless gauge-singlet composite spin-1/2 fermions [8–14].

2.2.3 Spontaneous Chiral Symmetry Breaking

Another possible phase in the strong-interaction region is the formation of bilinear fermion condensates. These bilinear fermion condensates partially or completely break global chiral symmetry, and also break chiral gauge symmetry if bilinear fermion condensates carry uncontracted gauge indices. Gauge bosons correspond to broken generators of gauge symmetry will be massive. Besides, there will be production of massless Nambu-Goldstone bosons correspond to broken generators of global flavor symmetry.

2.3 Gauge Anomalies in Chiral Gauge Theories

At the classical level, there is no restriction on the representations of chiral fermions in chiral gauge theories. However, at the quantum level, the axial vector (chiral) anomaly puts strong constraints on representations of chiral fermions. Consider the gauge symmetry current for left-handed fermions in the R representation:

$$j^{\mu,a} = \bar{\psi}_{i,L}\gamma^{\mu}t^{\alpha}\psi_{L,i} \tag{2.5}$$

where we omit the gauge group indices. The divergence of this gauge current is given by

$$\langle p,\nu,b;k,\lambda,c|\partial_{\mu}j^{\mu,a}|0\rangle = \frac{g^2}{8\pi^2}\epsilon^{\alpha\nu\beta\lambda}p_{\alpha}k_{\beta}\cdot\mathcal{A}^{abc}(R)\ , \tag{2.6}$$

where $\mathcal{A}^{abc}(R)$ is defined as

$$\mathcal{A}^{abc}(R) = \mathrm{Tr}_R(T_a,\{T_b,T_c\}) = \mathcal{A}(R)d_{abc}\ . \tag{2.7}$$

Eq.(2.6) implies that the gauge current $j^{\mu,a}$ is not conserved unless $\cdot\mathcal{A}^{abc}(R)$ vanishes. If such a violation of gauge current conservation occurred, it would spoil the renormalizability of the theory (e.g., [74]).

In order to avoid this problem in a chiral gauge theory, it is necessary either that the theory is intrinsically anomaly-free, i.e., has vanishing $\mathcal{A}^{abc}(R)$ for all R or that the fermion contributions cancel each other in their contribution to the overall anomaly. In either case, one says that the theory is anomaly-free. An example of an intrinsically anomaly-free chiral gauge theory is one with a gauge group SO$(4k+2)$ with $k \geq 2$ and fermions in the spinor representation (e.g., the SO(10) grand unified theory). For general chiral gauge theories with nonzero $\mathcal{A}^{abc}(R)$, the requirement that they must be anomaly-free gives non-trivial constraints on the representations and numbers of non-singlet fermions. For example, in a chiral gauge theory with n_S copies of fermion in symmetric rank-2 (S) representation and $n_{\bar{F}}$ copies of fermions in conjugate fundamental representation ($\bar{F}$), the requirement of a vanishing chiral anomaly requires that $n_{\bar{F}} = (N+4)n_S$, where we use the property that $\mathcal{A}(S) = -(N+4)\mathcal{A}(\bar{F})$.

Chapter 3

Theoretical Framework and Methods of Analysis

In this chapter we introduce some useful methods for constructing and analyzing chiral gauge theories.

3.1 Renormalization-Group Evolution and Beta Function

We discussed the beta function above and recall that

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \tag{3.1}$$

where we have extracted an overall minus sign, b_ℓ is the ℓ-loop coefficient, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. The n-loop beta function, denoted $\beta_{\alpha,n\ell}$, is given by Eq. (3.1) with the upper limit on the ℓ-loop summation equal to n instead of ∞. The property of asymptotic freedom means that $\beta_\alpha < 0$ for small α. With the minus sign extracted in the perturbative expansion (3.1), this is satisfied if $b_1 > 0$. The one-loop and two-loop coefficients b_1 [75] and b_2 [76] are independent of the scheme used for regularization and renormalization [77], while the b_ℓ with $\ell \geq 3$ are scheme-dependent.

In the analysis of phases of chiral gauge theories we first calculate the one-loop coefficient b_1 and two-loop coefficient b_2. For the theory with $b_2 >$ 0, we infer that it is likely to be strongly coupled in the IR limit. For the theory with $b_2 < 0$, we calculate the fixed-point value $\alpha_{IR,2\ell}$ at which two-loop beta function is zero, namely $\alpha_{IR,2\ell} = -4\pi b_1/b_2$, as given above. We then compare $\alpha_{IR,2\ell}$ to the estimated critical value for the formation of bilinear condensate $\alpha_{cr,Ch}$(see definition in Eq.(3.4)) in a give channel, Ch. If $\alpha_{IR,2\ell}$ is substantially less than $\alpha_{cr,Ch}$, then the coupling will remain as approximately $\alpha_{IR,2\ell}$ in the IR limit approximately. If, on the other hand, $\alpha_{IR,2\ell}$ is substantially larger than $\alpha_{cr,Ch}$, then the would-be IR zero of the beta function is not exact. Rather than evolving to this would-be IRFP, there will generically be fermion condensate formation at some scale μ_c. As a result, the fermions involved in the condensate will gain dynamical masses and be integrated out of the low-energy field theory that describes the physics for scales $\mu < \mu_c$. Therefore, the coefficients in the beta function will change. Indeed, in an irreducibly chiral gauge theory, this condensate breaks the gauge symmetry, so that the resultant subgroup gauge symmetry in this low-energy effective field theory will be different from that in the UV theory. The low-energy theory then generically evolves to further strong coupling as μ decreases further in the IR.

3.2 Most Attractive Channel (MAC) Criterion

In a theory whose UV to IR evolution leads to a gauge coupling that is strong enough to produce several different types of fermion condensates, one method that has been widely used to predict which type of condensate is most likely to form is the most-attractive-channel (MAC) approach [18]. Consider a condensation channel in which fermions in the representations R_1 and R_2 of a given gauge group form a condensate that transforms according to the representation $R_{cond.}$ of this group, denoted

$$R_1 \times R_2 \to R_{cond.} \ . \tag{3.2}$$

An approximate measure, based on one-gluon exchange, of the attractiveness of this condensation channel, is

$$\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{Ch}) \ , \tag{3.3}$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation R, and $R_{Ch} \equiv R_{cond.}$. At this level of one-gluon exchange, if ΔC_2 is positive (negative), then the channel is attractive (repulsive).

An analysis of the Schwinger-Dyson equation for the propagator of a massless fermion transforming according to the representation R of a gauge group G shows that, in the ladder (i.e., iterated one-gluon exchange) approximation the minimum value of α for which fermion condensation occurs in a vectorial gauge theory is given by the condition that $3\alpha_{cr}C_2(R)/\pi \sim 1$, or equivalently, $3\alpha_{cr}\Delta C_2/(2\pi) = 1$, since $\Delta C_2 = 2C_2(R)$ in this case [78]. Therefore, a rough estimate for the minimal value of the running coupling which is sufficient to cause condensation in a given channel Ch is

$$\alpha_{cr,Ch} \sim \frac{2\pi}{3\Delta C_2(R)_{Ch}} \ . \tag{3.4}$$

Because of the strong-coupling nature of the fermion condensation process, Eq. (3.4) is only a rough estimate. For the expression of Eq.(3.4), we can see that $\alpha_{cr,Ch}$ is proportional to the inverse of $\Delta C_2(R)_{Ch}$, which implies that during the UV to IR evolution, the channel that has the maximum value of $\Delta C_2(R)_{Ch}$ is most likely to occur. In other words, the most attractive channel is the one that yields the maximum (positive) value of $\Delta C_2(R)_{Ch}$. Therefore, the MAC approach predicts that if, *a priori*, several condensation channels could occur, then the one that actually occurs is the channel that has the largest (positive) value of ΔC_2. The MAC method was applied, for example, in efforts to build reasonably UV-complete models with dynamical electroweak symmetry breaking [20,21,24–26,79]. These models made use of asymptotically free chiral gauge interactions that became strongly coupled, naturally leading to the formation of certain condensates (of fermions subject to the chiral gauge interaction) in a hierarchy of scales corresponding, via inverse powers, to the observed generational hierarchy of Standard-Model fermion mass scales.

For chiral gauge theories developing a fixed point $\alpha_{IR,2\ell}$ in the two-loop beta function, a measure of the likelihood that the coupling grows large enough in the infrared to produce fermion condensation in a given channel Ch is thus the ratio

$$\rho_{_{IR,Ch}} \equiv \frac{\alpha_{IR,2\ell}}{\alpha_{cr,Ch}} \ . \tag{3.5}$$

If this ratio is significantly larger (smaller) than unity, one may infer that condensation in the channel Ch is likely (unlikely). As with the use of the MAC, this ρ test is only a rough estimate.

3.3 Global Anomalies Matching Condition

As mentioned above, in the strongly coupled region, instead of spontaneous chiral symmetry breaking with formation of bilinear fermion condensate, chiral gauge theories may evolve to an alternative phase, where the gauge interaction confines and produces massless gauge-singlet composite fermions. In this case, both chiral gauge symmetry and global chiral symmetry are unbroken. A necessary condition for the formation of these massless composite fermions is that the fermion content should satisfy the 't Hooft anomaly-matching condition [8].

The 't Hooft anomaly-matching condition states that anomaly of global symmetry calculated in the UV limit where fermionic degrees of freedom are elementary fermions should be equal to the global anomaly calculated in the IR limit where fermionic degrees of freedom are massless composite spin-1/2 fermions. The proof is as follows [8]: We add spectator gauge fields that turn the global symmetry into a local gauge symmetry. The associated coupling constants may all be arbitrarily small, so that the dynamics of the strong color gauge interactions is negligibly affected. For a consistent gauge interaction, we have to also add spectator fermions to cancel the anomaly of this artificial gauge symmetry:

$$\mathcal{A}(\text{UV fermions}) + \mathcal{A}(\text{spectator fermions}) = 0 \ , \tag{3.6}$$

where $\mathcal{A}(...)$ denote global anomaly of corresponding fermions. In the low energy IR theory (much lower than the binding scale of the original gauge interaction), there are massless composite fermions due to the strong interaction. These composite states form new representations of the gauged-global symmetry. The spectrum of spectator fermions is unchanged due to the smallness of the artificial gauge interaction. Since the condition of anomaly freedom should always be satisfied, we have

$$\mathcal{A}(\text{IR composite fermions}) + \mathcal{A}(\text{spectator fermions}) = 0 \ . \tag{3.7}$$

Combining Eq.(3.6) and Eq.(3.7), we then have

$$\mathcal{A}(\text{UV fermions}) = \mathcal{A}(\text{IR composite fermions}) \ . \tag{3.8}$$

This consistency condition can help to testify whether a strongly coupled chiral gauge theory can form massless composite fermions without chiral symmetry breaking.

3.4 Degree-of-Freedom Inequality

A quantity that can give predictions for renormalization-group evolution involves the relevant perturbative field degrees of freedom in the effective field theory that is applicable at a given reference scale, μ. From the study of second-order phase transitions and critical phenomena in statistical mechanics and condensed matter physics, one is familiar with the Wilsonian thinning of degrees of freedom as one changes the scale at which one measures physical quantities from short distances (UV) to large distances (IR). Given the correspondence between the inverse distance and the reference momentum scale μ, one may naturally expect a similar decrease (or non-increase) of dynamical degrees of freedom in a quantum field theory as μ decreases from large values in the ultraviolet to small values in the infrared [80].

Given that a theory is asymptotically free, the gauge coupling approaches zero in the deep ultraviolet as $\mu \to \infty$, so that one can identify and enumerate the perturbative degrees of freedom in the fields. Depending on the theory, it may also be true that in the deep infrared, as $\mu \to 0$, the residual (massless) particles are weakly interacting, so that again one can describe them perturbatively and enumerate their degrees of freedom. Although one is describing the UV to IR evolution of a zero-temperature quantum field theory, a natural approach to the enumeration of the perturbative degrees of freedom in the fields is provided by envisioning a finite-temperature field theory, where the temperature T corresponds to the Euclidean scale, μ, and using the count embodied in the free energy density, $F(T)$. This is given by

$$F(T) = f(T)\,\frac{\pi^2}{90}\,T^4 \tag{3.9}$$

with

$$f = 2N_v + \frac{7}{4}N_f + \frac{7}{8}N_{f,Maj} + N_s\ , \tag{3.10}$$

where N_v and N_s are the number of vector and (real) scalar fields, and N_f and $N_{f,Maj}$ are the number of chiral components of Dirac and Majorana fermions in the theory, respectively [81]. Assuming that the relevant fields become free in the respective UV and IR limits, we define

$$f_{UV} = f(\infty), \quad f_{IR} = f(0)\ . \tag{3.11}$$

Since the theories that we consider are required to be asymptotically free, we can always identify the Lagrangian fields in the deep UV and hence calculate f_{UV}.

In accord with experience in statistical mechanics, Ref. [84] conjectured the degree-of-freedom inequality

$$\Delta f \equiv f_{UV} - f_{IR} \geq 0 \tag{3.12}$$

for vectorial gauge theories, and Ref. [83] extended this conjecture to chiral gauge theories. In [83] this conjecture was applied to analyze several asymptotically free chiral gauge theories. Subsequent studies have investigated the possible types of IR behavior involving strong coupling and condensate formation; Refs. [22, 82] are particularly relevant for our current work.

As noted above, since we restrict to asymptotically free theories, the condition that the theory becomes free as $\mu \to \infty$ is always satisfied. There are three types of situations where the condition that the fields are also weakly coupled in the IR is satisfied. In all of these we can calculate f_{IR}. In the first of these, the theory evolves to an exact, weakly coupled IR fixed point, so that the field degrees of freedom in the massless fields are the same as they were in the UV, up to small, calculable perturbative corrections, which obey the inequality (3.12) [83, 84]. In the second type of situation, there is global and/or gauge symmetry breaking at one or more scales, so that as μ decreases below these scales toward the infrared, in the applicable low-energy effective field theory, the remaining massless particles are Nambu-Goldstone bosons (NGBs) resulting from the spontaneous chiral symmetry breaking. Since the NGBs have only derivative interactions among themselves, which vanish as $\sqrt{s}/\Lambda \to 0$, where $\sqrt{s}$ is the center-of-mass energy and Λ denotes the scale of chiral symmetry breaking, it follows that these NGBs become free in the infrared limit. A third type of possible situation is one in which the chiral gauge interaction confines and produces massless gauge-singlet composite fermions. The interactions between these gauge-singlet fermions involve higher-dimension operators and hence are also weak in the infrared. In some models, the second and third types of behavior can occur together [22].

A direct test of the conjectured degree-of-freedom inequality (3.12) for asymptotically free chiral gauge theories would probably require lattice simulations. However, because of fermion doubling on the lattice (in which a single continuum fermion produces 2^d fermion modes on a d-dimensional Euclidean lattice, with half corresponding to one sign of γ_5 and the other half corresponding to the opposite sign of γ_5), it has been challenging to simulate chiral gauge theories via lattice methods. A different approach to testing the validity of the conjecture is to study its application to vectorial gauge theories. These have the advantage that they can be simulated on the lattice, and there are well-understood ways of dealing with fermion doubling so that in the continuum limit one should be able to determine the actual number, N_f, of active fermions. Ongoing lattice studies of the infrared behavior of various vectorial gauge theories, such as a gauge theory with $G = \mathrm{SU}(2)$ and $N_f = 6$ Dirac fermions in the fundamental representation [85], are making progress in testing the conjectured degree-of-freedom inequality.

3.5 Low Energy Effective Field Theory

In the case of a strong gauge interaction that leads to formation of bilinear fermion condensates with spontaneous symmetry breaking at energy scale Λ_1, the gauge bosons corresponding to the broken gauge symmetry, as well as fermions involved in condensates, acquire dynamical masses of order Λ_1. At energy scales below Λ_2, we integrate out these massive degrees of freedom and construct an effective theory with the remaining massless fields that is applicable as the reference energy scale μ decreases below Λ_1. In general, in the residual effective field theory, there can be several such stages of condensate formation and symmetry breaking at different energy scale $\Lambda_2 > \Lambda_3 > ....$ So there is an associated sequence of effective field theories that describe the physics at different intervals of μ.

3.5.1 Anomaly Freedom of a Low-Energy Effective Theory Arising from Dynamical Breaking of a Chiral Gauge Theory

In general, a low-energy effective field theory that arises from an anomaly-free chiral gauge theory via dynamical gauge symmetry breaking is also anomaly-free. The proof is as follows [2]: Let us consider a chiral gauge theory with a gauge group G and an anomaly-free set of chiral fermions transforming according to some set of representations $\{R_i\}$ of G. Without loss of generality, we take all of the fermions to be left-handed. Also without loss of generality, we assume that this theory is irreducibly chiral, i.e., does not contain any vectorlike subsector. This assumption does not entail any loss of generality because the fermions in a vectorlike subsector give zero contribution to a chiral anomaly. Because the theory is irreducibly chiral, the gauge symmetry precludes any fermion mass terms in the fundamental lagrangian. To begin with, we assume that G is a simple group and discuss later the straightforward generalization of our argument to the case where G is a direct-product group. Let us denote the contribution of a chiral fermion in the R_i representation to the triangle anomaly in gauged currents as $\mathcal{A}(R_i)$. The property that the initial theory is anomaly-free is the condition

$$\sum_i n_{R_i} \mathcal{A}(R_i) = 0 \ , \tag{3.13}$$

where n_{R_i} denotes the number of copies of fermions in the representation R_i. This anomaly cancellation condition (3.13) also implies that if one restricts to a subgroup $H \subset G$, which means decomposing each representation R_i into representations R'_i of H, then the sum of contributions is also zero. Now, assume that this theory is asymptotically free, so that as the Euclidean reference scale μ decreases from the UV to the IR, the running gauge coupling increases, and assume further that this gauge coupling becomes strong enough at a scale Λ to produce bilinear fermion condensates that break the original gauge symmetry G to a subgroup $H \subset G$. The fermions involved in the condensate gain dynamical masses of order Λ, and the gauge bosons in the coset G/H also gain masses of this order.

To construct the low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ, one integrates out these massive states and enumerates the remaining H-nonsinglet massless fields. This enumeration involves decomposing each fermion representation R_i of G in terms of representations R'_i of H. The resultant anomaly cancellation condition in the low-energy effective theory that is the descendant of the original theory is

$$\sum_i n_{R'_i} \mathcal{A}(R'_i) = 0 \ , \tag{3.14}$$

where $\mathcal{A}(R'_i)$ refers to the contribution to the anomaly in the descendant theory from the fermions in the R'_i representation of the gauge group H. Now the fermions in the original theory that were involved in the condensate, and hence acquired dynamical masses and were integrated out, transform as singlets under H, and therefore, even if they were included in Eq. (3.14), they would make zero contribution to this sum. Combining this fact with Eq. (3.13), we deduce that the remaining H-nonsinglet fermions must also make zero net contribution in Eq. (3.14). This proves the theorem.

We make some further remarks on this result. In general, an asymptotically free chiral gauge theory that becomes strongly coupled and produces fermion condensates that dynamically break the gauge symmetry may undergo not just one, but several sequential stages of dynamical gauge symmetry breaking. Clearly, the theorem above applies not just to the first stage, but also to subsequent stages of symmetry breaking. As noted, it is straightforward to extend this theorem to the case where the gauge group of the theory is a direct-product group instead of a simple group. An example of this is given below in our analysis of the low-energy effective SU(5) $\otimes$ U(1) theory resulting as a descendant from an initial (anomaly-free) SU(6) chiral gauge with fermions in the S_2 and $\bar{A}_2$ representations of SU(6).

Chapter 4

Chiral Gauge Theories with Fermions in Fundamental, Symmetric Rank-2 and Higher-Dimensional Representations

In this and the later chapters, we proceed to discuss results on the UV to IR evolution and nonperturbative behavior of (anomaly-free) chiral gauge theories. These were published in collaboration with Prof. R. Shrock in the papers [1]- [4].

In the present chapter we calculate the UV to IR evolution and analyze possible types of infrared behavior for several asymptotically free chiral gauge theories with gauge group SU(N) and massless chiral fermions transforming according to a symmetric rank-2 tensor representation S and $N+4$ copies (flavors) of a conjugate fundamental representation $\bar{F}$, together with a vectorlike subsector with chiral fermions in higher-dimensional representation(s).

We achieve the goal of constructing chiral gauge theories where several methods that one can use to investigate the ultraviolet to infrared evolution of a chiral gauge theory give consistent results. These include (i) (perturbative) calculation of the beta function and analysis of possible IR zeros of this beta function; (ii) use of the most-attractive-channel (MAC) approach, which can suggest in which channel(s) bilinear fermion condensates are most likely to form [18] if the coupling gets sufficiently strong in the infrared; and (iii) a conjectured inequality involving the perturbative degrees of freedom in the massless fields [83,84]. We will denote this as the conjectured DFI, where DFI stands for degree of freedom inequality. In particular, in our constructions of chiral gauge theories, the expected type(s) of UV to IR evolution obey the conjectured degree-of-freedom inequality throughout the full range of parameters specifying the fermion contents of these theories. Our analysis extends the published results in [22].

4.1 Strategy for Construction of New Chiral Gauge Theories

Our general method for constructing the chiral gauge theories presented here is as follows. We take the gauge group to be $G = \mathrm{SU}(N)$ and include, as the irreducibly chiral sector of the theory, fermions transforming as the S and $(N+4)$ copies of $\bar{F}$. We choose the vectorlike subsector to consist of p copies of fermions that transform according to representation(s) R of G such that the channel

$$R \times \bar{R} \to 1 \tag{4.1}$$

is more attractive than other channels. (For some of our theories, $R = \bar{R}$.) In the theories that we consider, the next-most-attractive channel is

$$S \times \bar{F} \to F \ . \tag{4.2}$$

The ΔC_2 attractiveness measures for these channels are

$$\Delta C_2 = 2C_2(R) \quad \text{for } R \times \bar{R} \to 1 \tag{4.3}$$

and

$$\Delta C_2 = C_2(S) = \frac{(N+2)(N-1)}{N} \quad \text{for } S \times \bar{F} \to \bar{F} \ , \tag{4.4}$$

so the condition that the $R \times \bar{R} \to 1$ channel is more attractive than the $S \times \bar{F} \to \bar{F}$ channel is that

$$\Delta C_2(R) = 2C_2(R) > \frac{(N+2)(N-1)}{N} \ . \tag{4.5}$$

In all the cases that we consider, this guarantees that the $R \times \bar{R} \to 1$ channel is the most attractive channel in which condensation thus occurs first as the theory evolves from the UV to the IR. Consequently, if the fermion content is such that the running coupling $\alpha(\mu)$ becomes sufficiently large in the infrared, then, because the MAC is (4.1), the fermion condensation at the highest energy scale occurs among the fermions in the vectorlike subsector of the model, via the channel $R \times \bar{R} \to 1$. The resultant low-energy effective field theory applicable below this scale is thus comprised of the irreducible chiral sector of the theory, equivalent to the $p = 0$ special case of the full theory, with just the S fermion and the $N + 4$ copies of the $\bar{F}$ fermion. The various possible types of UV to IR evolution of this $p = 0$ theory obey the conjectured degree-of-freedom inequality [22, 82, 83].

4.2 Theory with $R = Adj$

4.2.1 Particle Content

In this section we construct and study a chiral gauge theory with gauge group SU(N) and fermion content consisting of chiral fermions transforming according to

1. a symmetric rank-2 tensor representation, S, with corresponding field $\psi_L^{ab} = \psi_L^{ba}$,
2. $N + 4$ copies (also called "flavors") of chiral fermions in the conjugate fundamental representation, $\bar{F}$, with fields $\chi_{a,i,L}$, $i = 1, ..., N + 4$, and
3. p copies of chiral fermions in the adjoint representation, denoted Adj, with fields $\xi^a_{b,j,L}$, $j = 1, ..., p$.

Here and below, $a, b, c...$ are gauge indices and i, j are copy indices. We call this the Adj theory by reference to the choice of the representation $R = R_{sc}$ for the fermions in the vectorlike subsector. This fermion content is summarized in Table 4.1. As noted above, we restrict to $N \geq 3$ because SU(2) has only (pseudo)real representations and hence a gauge theory based on the gauge group SU(2) is not chiral. This theory thus depends on the two integer parameters, $N \geq 3$ and $p \geq 0$, with an upper limit on p given by Eq. (4.10) below. We will sometimes use the Young tableaux $\square\square$ and $\bar{\square}$ for S and $\bar{F}$. The irreducibly chiral sector of this theory is comprised of the S and the $N + 4$ copies of $\bar{F}$ fermions, and the vectorlike subsector is comprised of the Adj fermions. Because of this self-conjugate nature of R_{sc}, the Adj

Table 4.1: Properties of fermions in the chiral gauge theories with vectorlike subsector consisting of p copies of fermions in the self-conjugate representation $R = R_{sc}$. The entries in the columns are: (i) fermion, (ii) representation of the SU(N) gauge group, (iii) number of copies, and representations (charges for abelian factors) of the respective factor groups in the global flavor symmetry group: (iv) $\mathrm{SU}(N+4)_{\bar{F}}$, (v) $\mathrm{SU}(p)_{R_{sc}}$, (vi) $\mathrm{U}(1)_1$, (vii) $\mathrm{U}(1)_2$. The notation for the fermion ξ in the R_{sc} is generic; specifically, this is $\xi^a_{b,i,L}$ for the *Adj* model and $\xi^{a_1,\ldots,a_k}_{i,L}$ for the AT model (with $N = 2k$). See text for further discussion.

fermion	SU(N)	no. copies	$\mathrm{SU}(N+4)_{\bar{F}}$	$\mathrm{SU}(p)_{R_{sc}}$	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$
$S:\ \psi^{ab}_L$	$\square\square$	1	1	1	$N+4$	$2pT_{R_{sc}}$
$\bar{F}:\ \chi_{a,i,L}$	$\bar{\square}$	$N+4$	$\square$	1	$-(N+2)$	0
$R_{sc}:\ \xi_L$	R_{sc}	p	1	$\square$	0	$-(N+2)$

fermions may be considered to be Majorana. Thus, if one were to remove the irreducibly chiral part of this theory and consider the part containing the gauge fields and the *Adj* fermions alone, the dynamical particle content in the Lagrangian would be analagous to the gluons and gluinos of an $\mathcal{N} = 1$ supersymmetric SU(N) gauge theory.

We recall that since the contribution to the triangle anomaly from S satisfies [86]

$$\mathrm{Anom}(S) = (N+4)\,\mathrm{Anom}(F) \ , \tag{4.6}$$

and since

$$\mathrm{Anom}(R) = -\mathrm{Anom}(\bar{R}) \ , \tag{4.7}$$

it follows that the set of chiral fermions S plus $(N+4)$ copies of $\bar{F}$ yields a theory that is free of anomalies in gauged currents. Furthermore, from Eq. (4.7), it follows that for any self-conjugate representation R_{sc}, $Anom(R_{sc}) = 0$. Hence, we are free to add fermions transforming according to a self-conjugate representation to a chiral gauge theory that is free of anomalies in gauged currents and it will retain this anomaly-free property. We use this fact here with $R_{sc} = Adj$.

4.2.2 Beta Function

The beta function for this *Adj* theory is given by Eq. (3.1) with the one-loop coefficient

$$(b_1)_{Adj} = \frac{1}{3}\Big[(9-2p)N - 6\Big] \tag{4.8}$$

and the two-loop coefficient

$$(b_2)_{Adj} = \frac{1}{6}\Big[(39-32p)N^2 - 90N + 3 + 36N^{-1}\Big] \ . \tag{4.9}$$

(See Appendix A for general formulas for b_1 and b_2.) These coefficients contain the maximal scheme-independent information about the dependence of the gauge coupling on the reference scale, μ. This information will suffice for our present purposes. Higher-loop effects for vectorial theories and effects of scheme transformations on higher-loop terms in the beta function for gauge theories have been studied in [87]- [99].

We denote the values of p for which $(b_1)_{Adj} = 0$ as $p_{b1z,Adj}$ (where the subscript stands for b_1 zero). This value is [100]

$$p_{b1z,Adj} = \frac{3(3N-2)}{2N} \ . \tag{4.10}$$

Our requirement that the model should be asymptotically free means that $\beta_\alpha < 0$ for small α. This is equivalent to the condition that $b_1 > 0$ or, if b_1 vanishes, then the further requirement that $b_2 > 0$. Now $(b_1)_{Adj} > 0$ if and only if $p < p_{b1z,Adj}$, i.e.,

$$p < \frac{3(3N-2)}{2N} \ . \tag{4.11}$$

This means that the set of physical, integral values of p allowed by our requirement of asymptotic freedom are $0 \le p \le 3$ for $N = 3, 4, 5, 6$ and $0 \le p \le 4$ for $N \ge 7$. Note that if $N = 6$ and $p = 4$, then $b_1 = 0$, so one must examine the sign of b_2 to determine if the theory is asymptotically free or not, and for this case $(b_2)_{Adj}$ is negative, hence excluding it from consideration. Here and below, for a given theory and value of N, we will denote the maximum allowed value of p as p_{max}.

As a consequence of the asymptotic freedom of the theory, the beta function always has a zero at $\alpha = 0$, which is a UV fixed point (UVFP) of the renormalization group. In general, the two-loop beta function, $\beta_{\alpha,2\ell}$, has an IR zero if b_2 has a sign opposite to that of b_1, i.e., if b_2 is negative. For $p = 0$,

$(b_2)_{Adj} > 0$, so $\beta_{\alpha,2\ell}$ has no IR zero. As p increases, $(b_2)_{Adj}$ decreases and eventually passes through zero to negative values, giving rise to an IR zero of $\beta_{\alpha,2\ell,Adj}$. Let us denote the value of p where b_2 vanishes as $p_{b2z,Adj}$. This is

$$p_{b2z,Adj} = \frac{3(13N^3 - 30N^2 + N + 12)}{32N^3} \ . \qquad (4.12)$$

In Table 4.2 we list values of $p_{b1z,Adj}$ and $p_{b2z,Adj}$ for this theory. The value $p_{b2z,Adj}$ is less than the upper bound on p, $p_{b1z,Adj}$, i.e.,

$$p_{b2z,Adj} < p_{b1z,Adj} \ . \qquad (4.13)$$

This inequality is proved by analyzing the difference,

$$p_{b1z,Adj} - p_{b2z,Adj} = \frac{3(35N^3 - 2N^2 - N - 12)}{32N^3} \ . \qquad (4.14)$$

This difference is positive for all physical N. Hence, for p in the interval [100]

$$(I_p)_{Adj} : \quad p_{b2z,Adj} < p < p_{b1z,Adj} \ , \qquad (4.15)$$

this theory is asymptotically free, and $\beta_{\alpha,2\ell,Adj}$ has an IR zero. The actual physical, integral values of p in the interval $(I_p)_{Adj}$ depend on the value of N. There are several different sets of N and p values where this IR zero is physical:

$$(I_p)_{Adj} : \qquad 1 \le p \le 3 \quad \text{if } 3 \le N \le 6,$$
$$1 \le p \le 4 \quad \text{if } 7 \le N \le 12,$$
$$2 \le p \le 4 \quad \text{if } N \ge 13 \ . \qquad (4.16)$$

These different cases follow from two properties. First, $p_{b1z,Adj}$ (understood to be generalized from positive integers to positive real numbers) is a monotonically increasing function of N for physical N and ascends through the value 4 as N increases through the value $N = 6$. Second, for $N > (1+\sqrt{1081}\,)/30 = 1.129$ and hence for the range $N \ge 3$ relevant here, $p_{b2z,Adj}$ is a monotonically increasing function and increases through 1 at $N = 12.7922$ (the largest root of $7N^3 - 90N^2 + 3N + 36$). Hence, if $N \ge 13$, the lowest value of $p \in (I_p)_{Adj}$ is $p = 2$, as indicated in (4.16).

For values of N and p where $\beta_{\alpha,2\ell,Adj}$ has a physical IR zero, it occurs at

$$\alpha_{IR,2\ell,Adj} \equiv 4\pi a_{IR,2\ell,Adj} = -4\pi \frac{(b_1)_{Adj}}{(b_2)_{Adj}}$$

$$= \frac{8\pi N[(9-2p)N-6]}{(32p-39)N^3+90N^2-3N-36} . \qquad (4.17)$$

In using this result, it should be recalled that, in general, an IR zero of a beta function at $\alpha_{IR,2\ell} = -4\pi b_1/b_2$ can be reliable if $|b_2|$ is not too small, i.e., when $\alpha_{IR,2\ell}$ is not too large for the perturbative calculation to be applicable. In Table 4.3 we list values of $\alpha_{IR,2\ell,Adj}$.

It is of interest to consider the limit [101]

$$N \to \infty \text{ with } \zeta(\mu) \equiv \alpha(\mu)N \text{ finite and } p \text{ fixed.} \qquad (4.18)$$

In this limit,

$$\lim_{N\to\infty} p_{b1z,Adj} = \frac{9}{2} \qquad (4.19)$$

and

$$\lim_{N\to\infty} p_{b2z,Adj} = \frac{39}{32} = 1.21875 , \qquad (4.20)$$

so that the interval $(I_p)_{Adj}$ becomes

$$\lim_{N\to\infty} (I_p)_{Adj} : \ \frac{39}{32} < p < \frac{9}{2} , \qquad (4.21)$$

containing the physical, integral values $p = 2, \ 3, \ 4$. In the large-N limit (4.18), the combination of α, or equivalently, a, and N that remains finite is

$$\zeta \equiv \lim_{N\to\infty} \alpha N . \qquad (4.22)$$

Correspondingly, the rescaled beta function that is finite has the form

$$\beta_\zeta \equiv \frac{d\zeta}{dt} . \qquad (4.23)$$

where, as in Eq. (2.3), $t = \ln\mu$. In this limit, for physical $p \in (I_p)_{Adj}$, the (rescaled, finite) $\beta_{\zeta,2\ell}$, has an IR zero at

$$\zeta_{IR,2\ell,Adj} = \frac{8\pi(9-2p)}{32p-39} . \qquad (4.24)$$

Table 4.2: Values of $p_{b1z,Adj}$ and $p_{b2z,Adj}$ in the Adj theory as functions of N.

N	$p_{b2z,Adj}$	$p_{b1z,Adj}$
3	0.3333	3.5000
4	0.5391	3.7500
5	0.6690	3.9000
6	0.7578	4.0000
7	0.8222	4.0714
8	0.8708	4.1250
9	0.90895	4.1667
10	0.9396	4.2000
11	0.9647	4.2773
12	0.9857	4.2500
13	1.0035	4.2692
14	1.0187	4.2857
15	1.0320	4.3000
10^2	1.1906	4.4700
10^3	1.2159	4.4970
∞	1.21875	4.5000

The approach to this limit of $N \to \infty$ involves correction terms that are powers in $1/N$:

$$N\alpha_{IR,2\ell,Adj} = \frac{8\pi(9-2p)}{32p-39} - \frac{96\pi(p+48)}{(32p-39)^2 N} + O\left(\frac{1}{N^2}\right) . \tag{4.25}$$

One may compare the approach to the $N \to \infty$ limit here with that in a (vectorial) SU(N) gauge theory with N_f fermions in the fundamental representation in the limit $N \to \infty$, $N_f \to \infty$ with the ratio N_f/N fixed and finite (and $\alpha(\mu)N$ a finite function of μ), denoted the LNN limit in [89]. In that case [88,89] the leading correction term to the limit was suppressed like $1/N^2$ instead of $1/N$, and the correction terms formed a series in powers of $1/N^2$ instead of powers in $1/N$. Hence, the approach to the $N \to \infty$ limit here is not as rapid as in the LNN limit.

Table 4.3: Values of $\alpha_{IR,2\ell,Adj}$ and $\rho_{IR,Adj\times Adj}$ in the *Adj* theory for an illustrative range of values of N and, for each N, the values of p in the respective interval $(I_p)_{Adj}$.

N	p	$\alpha_{IR,2\ell,Adj}$	$\rho_{IR,Adj\times Adj}$
3	1	1.96	5.63
3	2	0.471	1.35
3	3	0.0982	0.281
4	1	2.34	8.95
4	2	0.470	1.80
4	3	0.120	0.457
5	1	2.75	13.1
5	2	0.448	2.14
5	3	0.121	0.579
6	1	3.24	18.6
6	2	0.4215	2.415
6	3	0.117	0.669
7	1	3.88	25.9
7	2	0.395	2.64
7	3	0.110	0.738
7	4	0.00504	0.0337
8	1	4.75	36.3
8	2	0.370	2.82
8	3	0.104	0.793
8	4	0.00784	0.0599
13	2	0.275	3.42
13	3	0.0768	0.954
13	4	0.0109	0.135
14	2	0.261	3.49
14	3	0.0728	0.973
14	4	0.01075	0.144
15	2	0.249	3.56
15	3	0.0692	0.991
15	4	0.0106	0.152

4.2.3 Analysis of UV to IR Flows

Because of the asymptotic freedom of the theory, i.e., the fact that the beta function is negative for small α, it follows that, as the Euclidean reference momentum scale μ decreases from the ultraviolet toward the infrared, $\alpha(\mu)$ increases. There are several possibilities for the behavior that can occur:

1. First, if the beta function has an IR zero at a sufficiently small value of $\alpha = \alpha_{_{IR}}$, then one expects that the theory will evolve into the infrared without any spontaneous chiral symmetry breaking. In this case, the IR zero of β_α is an exact IRFP of the renormalization group, so that as $\mu \to 0$, the theory exhibits scale invariance with nonzero anomalous dimensions. In the IR limit $\mu \to 0$, one anticipates that the theory is in a deconfined, massless non-Abelian Coulomb phase.

2. For smaller values of p, the IR zero of the beta function is larger, and correspondingly, $\alpha(\mu)$ becomes larger as μ decreases from the UV to the IR. Then the strongly coupled gauge interaction can produce fermion condensates that break global and possibly also local gauge symmetries. This behavior also applies if p is sufficiently small that the beta function has no IR zero, so that $\alpha(\mu)$ keeps increasing with decreasing μ until it exceeds the interval where the perturbative beta function describes its evolution. In this general category of UV to IR evolution, there can be a sequence of condensate formations at various energy scales.

3. In the strongly coupled case (including both the subcases where the beta function has an IR zero at sufficiently large coupling and where the beta function has no IR zero), an alternate possibility is, if the fermion content satisfies the 't Hooft anomaly-matching conditions [8], then the gauge interaction might confine and produce massless gauge-singlet composite fermions.

The beta function describes the growth of $\alpha(\mu)$ as the reference momentum scale μ decreases from the UV to the IR. If the fermion content is such that the beta function has no IR zero, then the interaction definitely becomes strongly coupled in the infrared. If, on the other hand, the beta function does have an IR zero, then one must investigate how large the value of the coupling is at this zero. In conjunction with knowledge of the probable channel in which fermions may condense and the corresponding estimate of the minimum critical coupling, α_{cr} that triggers this condensation, one can then draw a plausible inference as to whether the condensation takes place or whether, in contrast, the theory evolves into the infrared without any fermion condensation or associated spontaneous chiral symmetry breaking.

The only composite fermions that one can form are those of the $p = 0$ theory, and we find that these do not match the global anomalies of $G_{fl,Rsc}$ (given below in Eq. (4.61) for $R_{sc} = Adj$. This rules out the possibility that the original theory can form massless composite fermions involving the full set of massless fermions in the theory with $p > 0$. As we will discuss below, however, if the UV to IR evolution leads to sufficiently strong coupling so that there is condensation in the $R_{sc} \times R_{sc} \to 1$ channel, giving the R_{sc} fermions dynamical masses, then in the low-energy effective field theory below the condensation scale, with these fermions removed, the descendant theory is equivalent to the original theory with $p = 0$. In this descendant theory (called the $S\bar{F}$ theory below), further evolution into the infrared might produce massless gauge-singlet composite fermions.

To obtain information concerning the likely type of UV to IR evolution among types 1 and 2 in the list above, as a function of p, we first identify the most attractive channel, which is

$$Adj \times Adj \to 1 \ . \tag{4.26}$$

This clearly preserves the SU(N) gauge symmetry, and has attractiveness measure

$$\Delta C_2 = 2N \quad \text{for } Adj \times Adj \to 1 \ . \tag{4.27}$$

In particular, this channel is more attractive than the $S \times \bar{F} \to F$ channel, in accordance with the inequality (4.5). Quantitatively, the difference in ΔC_2 values for these two channels is

$$\begin{aligned} & \Delta C_2(Adj \times Adj \to 1) - \Delta C_2(S \times \bar{F} \to F) \\ = \ & \frac{N^2 - N + 2}{N} \ , \end{aligned} \tag{4.28}$$

which is positive for all physical N. The condensates for the $Adj \times Adj \to 1$ channel are

$$\langle \xi^{a\ T}_{b,i,L} C \, \xi^b_{a,j,L} \rangle \ , \quad i, \ j = 1, ..., p \ . \tag{4.29}$$

From Eq. (4.27), we obtain the rough estimate of the minimal critical coupling for condensation in the $Adj \times Adj \to 1$ channel:

$$\alpha_{cr,Adj\times Adj} \simeq \frac{\pi}{3N} \ . \tag{4.30}$$

Thus, an approximate indication of the size of the IR fixed point relative to the size that would lead to the formation of fermion condensates in the channel Eq. (4.26) is the ratio

$$\rho_{IR,Adj\times Adj} \equiv \frac{\alpha_{IR,2\ell,Adj}}{\alpha_{cr,Adj\times Adj}} = \frac{24N^2[(9-2p)N-6]}{(32p-39)N^3+90N^2-3N-36} \ . \tag{4.31}$$

As p decreases, $\alpha_{IR,2\ell}$ increases. Therefore, considering N and p as being extended from the non-negative integers to the non-negative real numbers, one can calculate a rough estimate of the critical value of p, denoted $p_{cr,Adj\times Adj}$, such that, as p decreases through this value, $\alpha_{IR,2\ell}$ increases through the value $\alpha_{cr,Adj\times Adj}$. This critical value of $p_{cr,Adj\times Adj}$ is thus obtained by setting $\rho_{IR,Adj\times Adj} = 1$ and solving for p, yielding

$$p_{cr,Adj\times Adj} \simeq \frac{3(85N^3-78N^2+N+12)}{80N^3} \ . \tag{4.32}$$

This critical value $p_{cr,Adj}$ is a monotonically increasing function of N for physical N, increasing from $67/30 = 2.23$ for $N = 3$ and, as $N \to \infty$,

$$\lim_{N\to\infty} p_{cr,Adj\times Adj} = \frac{51}{16} = 3.1875 \ , \tag{4.33}$$

where the limit is approach from below as N increases.

We list values of the ratio $\rho_{IR,Adj\times Adj}$ in Table 4.3 for several illustrative values of N and p. For all of the values of N presented in this table, the respective values of the ratio $\rho_{IR,Adj\times Adj}$ for $p = 4$ are much smaller than 1, so that one can conclude that for $p = 4$, the theory evolves from the UV to a scale-invariant, non-Abelian Coulomb phase in the IR. As is evident from Table 4.3, for a given N, as p decreases, $\alpha_{IR,2\ell,Adj}$ increases. As this IR coupling becomes of O(1), the uncertainties in the use of perturbation theory increase. For most of $p = 3$ cases shown with various N, the ratio $\rho_{IR,Adj\times Adj}$ is sufficiently close to 1 that, taking account of these uncertainties, one cannot draw a definite conclusion as to whether fermion condensate does or does not take place. For the cases shown in Table 4.3 with $p = 1$ (where this is in $(I_p)_{Adj}$) and $p=2$, the ratio $\rho_{_{IR,Adj\times Adj}}$ is substantially larger than 1, so that in these cases, one expects that the gauge interaction become strong enough to produce fermion condensation in the channel (4.26).

In the large-N limit defined above,

$$\lim_{N\to\infty}\rho_{_{IR,Adj\times Adj}} = \frac{24(9-2p)}{32p-39} \ . \tag{4.34}$$

In particular,

$$\lim_{N\to\infty}\rho_{_{IR,Adj\times Adj}} = \frac{24}{89} = 0.270 \quad \text{for } p=4 \tag{4.35}$$

$$\lim_{N\to\infty}\rho_{_{IR,Adj\times Adj}} = \frac{72}{57} = 1.26 \quad \text{for } p=3 \tag{4.36}$$

$$\lim_{N\to\infty}\rho_{_{IR,Adj\times Adj}} = \frac{24}{5} = 4.80 \quad \text{for } p=2 \tag{4.37}$$

(where the floating-point results are given to the indicated accuracy). Hence, in this large-N limit, since the limit of the ratio $\rho_{_{IR,Adj\times Adj}}$ for $p=4$ is sufficiently small compared to 1 that it is plausible that in the IR the theory is in a deconfined Coulombic phase, while if $p=3$, $\rho_{_{IR,Adj\times Adj}}$ is too close to unity for one to be able to draw a definite conclusion. Finally, if $p=2$, then $\rho_{_{IR,Adj\times Adj}}$ is sufficiently large compared with 1 that one expects that the theory can produce bilinear condensates in the most attractive channel, as discussed above.

We continue with the analysis of the UV to IR evolution for the smaller values of p that produce a strongly coupled gauge interaction. As the momentum scale μ decreases through a scale denoted Λ_{Adj}, $\alpha(\mu)$ exceeds $\alpha_{cr,Adj}$, and, from our discussion above, we infer that the gauge interaction produces the bilinear fermion condensates (4.29) in the MAC, $Adj \times Adj \to 1$. These condensates preserve the SU(N) gauge symmetry and the $\mathrm{U}(1)_1$ global symmetry, while breaking the $\mathrm{U}(1)_2$ and SU(p) global symmetries (these global symmetries are defined in Sect. 4.4). By the use of a vacuum alignment argument [102], one can plausibly infer that the condensates (4.29) have $i=j$, with $i=1,...,p$ and hence preserve an SO(p) global isospin symmetry defined by the transformation

$$\xi^a_{b,i,L} \to \sum_{j=1}^{p} \mathcal{O}_{ij}\xi^a_{b,j,L} \ , \quad \mathcal{O}\in \mathrm{SO}(p) \ . \tag{4.38}$$

Just as light quarks gain dynamical, constituent quark masses of order Λ_{QCD} due to the formation of $\langle \bar{q}q \rangle$ condensates in quantum chromodynamics (QCD), so also, the $p(N^2-1)$ components, $\xi^a_{b,i,L}$, of the Adj fermions involved in these condensates pick up a common dynamical mass of order Λ_{Adj}.

At scales $\mu < \Lambda_{Adj}$, the analysis proceeds by integrating out the massive $\xi^a_{b,j,L}$ fermions, constructing the low-energy effective field theory applicable for these lower scales, and then exploring the further evolution of this descendant theory into the infrared. Since the condensation (4.29) gives dynamical masses to all of the Adj fermions $\xi^a_{b,j,L}$, $j = 1, ..., p$, the low-energy effective theory below this condensation scale Λ_{Adj} is just the $p = 0$ theory. Since the evolution of this theory is the same as for our second type of chiral gauge theory, we first study this second theory, and then discuss the further IR evolution.

4.3 Theory with $N = 2k$ and $R = [N/2]_N$

4.3.1 Particle Content

In this section we construct and study a chiral gauge theory with gauge group $G = \mathrm{SU}(N)$ with even $N = 2k$, and fermions transforming according to

1. a symmetric rank-2 tensor representation, S, with corresponding field $\psi^{ab}_L = \psi^{ba}_L$,
2. $N + 4$ copies chiral fermions in the conjugate fundamental representation, $\bar{\square}$, with fields $\chi_{a,i,L}$, $i = 1, ..., N + 4$, and
3. p copies of chiral fermions in the totally antisymmetric k-fold tensor representation $[N/2]_N = [k]_{2k}$, with fields $\xi^{a_1 \ldots a_k}_{j,L}$, $j = 1, ..., p$.

We again label this theory by the representation of the fermions in the vectorlike subsector, namely AT, for antisymmetric k-fold tensor. This fermion content is summarized in Table 4.1.

The representation $[k]_N$ has the dimension (for general N)

$$\dim([k]_N) = \binom{N}{k} \tag{4.39}$$

and satisfies the equivalence property

$$[N-k]_N = \overline{[k]}_N \ . \tag{4.40}$$

Here we have used the standard notation for the binomial coefficient, $\binom{a}{b} \equiv a!/[b!\,(a-b)!]$. An important property that follows from Eq. (4.40) that that we will use here is the fact that for our case of interest, $N = 2k$, the representation $[k]_{2k}$ is self-conjugate:

$$[k]_{2k} = \overline{[k]}_{2k} \ . \tag{4.41}$$

Combining the self-conjugate property of $[N/2]_N = [k]_{2k}$ with the relation (4.7), it follows that

$$\text{Anom}([k]_{2k}) = 0 \ . \tag{4.42}$$

Thus, this theory has the same irreducibly chiral sector as the theory discussed in the previous section, and a vectorlike subsector that consists of the p copies of the fermions in the $[N/2]_N$ representation.

4.3.2 Beta Function

We calculate that the one- and two-loop terms in the beta function of this theory are, in terms of $k = N/2$,

$$(b_1)_{AT} = 6k - 2 - \frac{p\,(2k-2)!}{3[(k-1)!]^2} \tag{4.43}$$

and

$$(b_2)_{AT} = \frac{52k^3 - 60k^2 + k + 6}{2k} - \frac{p\,k(43+6k)\,(2k-2)!}{12[(k-1)!]^2} \ . \tag{4.44}$$

For small p, $(b_1)_{AT}$ is positive, and as p increases, $(b_1)_{AT}$ decreases and passes through zero as p exceeds the value

$$p_{b1z,AT} = \frac{6(3k-1)[(k-1)!]^2}{(2k-2)!} \ . \tag{4.45}$$

The requirement that the theory should be asymptotically free is thus satisfied if

$$p < \frac{6(3k-1)[(k-1)!]^2}{(2k-2)!} \ . \tag{4.46}$$

This upper bound decreases rapidly as a function of $k = N/2$, so that as k increases, eventually the requirement of asymptotic freedom precludes any nonzero value of p. Thus, the AT theory has no asymptotically free large-N

limit with nonzero p, in contrast to the Adj theories constructed and studied here.

The beta function of the AT theory has an IR zero if b_2 is negative. For small p, $(b_2)_{AT}$ is positive, and it decreases through zero to negative values as p (continued to the real numbers) increases through the value

$$p_{b2z,AT} = \frac{6(52k^3 - 60k^2 + k + 6)\,[(k-1)!]^2}{k^2(6k+43)\,(2k-2)!} \ . \tag{4.47}$$

We observe that $p_{b1z,AT} > p_{b2z,AT}$. This is proved by considering the difference,

$$p_{b1z,AT} - p_{b2z,AT} = \frac{6(18k^4 + 71k^3 + 17k^2 - k - 6)[(k-1)!]^2}{k^2(43+6k)[(2k-2)!]} \ . \tag{4.48}$$

This difference is positive for all k values of relevance here (with k extended to the positive reals, it is positive for $k > 0.3724$). By itself, this inequality does not guarantee that there is an integral value of p that lies above $p_{b2z,AT}$ and below $p_{b1z,AT}$, but in fact we find that for each relevant case, there are one or more such integral values. These then define the respective intervals $(I_p)_{AT}$,

$$(I_p)_{AT}: \quad p_{b2z,AT} < p < p_{b1z,AT} \tag{4.49}$$

for each k. For the (integral) values of $p \in (I_p)_{AT}$, the beta function of the SU($2k$) AT theory has an IR zero. We list the values of p_{b1z}, p_{b2z}, p_{max}, and $(I_p)_{AT}$ in Table 4.5). Note that for the cases $G = \mathrm{SU}(N)$ with $k \geq 2$ under consideration here, the requirement of asymptotic freedom allows nonzero values of p only for $k \leq 5$.

For a given $N = 2k$ whith a nonvacuous interval $(I_p)_{AT}$, the $\beta_{\alpha,2\ell}$ has an IR zero at

$$\alpha_{IR,2\ell,AT} = -\frac{4\pi(b_1)_{AT}}{(b_2)_{AT}} \tag{4.50}$$

where $(b_1)_{AT}$ and $(b_2)_{AT}$ were given in Eqs. (4.43) and (4.44) above. We list the values of $\alpha_{IR,2\ell,AT}$ in Table 4.4.

4.3.3 UV to IR Evolution

Here we analyze the UV to IR evolution of this AT chiral gauge theory. By construction, the most attractive channel involves fermion condensation in

Table 4.4: Values of $\alpha_{IR,2\ell,AT}$ and $\rho_{IR,AT}$ in the AT theory for the relevant values of N and, for each N, the values of p in the respective interval $(I_p)_{AT}$.

N	p	$\alpha_{IR,2\ell,AT}$	$\rho_{IR,AT}$
4	3	11.170	26.67
4	4	3.371	8.05
4	5	1.8345	4.38
4	6	1.178	2.81
4	7	0.814	1.94
4	8	0.583	1.39
4	9	0.422	1.01
4	10	0.305	0.728
4	11	0.215	0.514
4	12	0.144	0.345
4	13	0.0871	0.208
4	14	0.0398	0.095
6	2	4.021	20.16
6	3	0.974	4.88
6	4	0.460	2.29
6	5	0.242	1.21
6	6	0.125	0.625
6	7	0.0508	0.255
8	1	1.290	11.08
8	2	0.183	1.57
8	3	0.0241	0.207
10	1	0.0360	0.473

the channel (4.1), with $R = [N/2]_N = [k]_{2k}$ in this case, i.e.,

$$[N/2]_N \times [N/2]_N \to 1 \ . \tag{4.51}$$

This preserves the SU(N) gauge symmetry and has the attractiveness measure

$$\Delta C_2 = 2C_2([N/2]_N) = \frac{k(2k+1)}{2} \ , \tag{4.52}$$

where we have used the result for $C_2([k]_N)$ given in Appendix A. The condensates are

$$\langle \epsilon_{a_1,...a_{2k}} \, \xi^{a_1,...,a_k \ T}_{i,L} C \, \xi^{a_{k+1},...,a_{2k}}_{j,L} \rangle \ , \ i,j = 1,...,p \ . \tag{4.53}$$

By a vacuum alignment argument, one may infer that these condensates have $i = j$ [102]. To show that the channel (4.51) is more attractive than the next-most-attractive channel, $S \times \bar{F} \to F$, we examine the difference

$$\begin{aligned} & \Delta C_2([N/2]_N \times [N/2]_N \to 1) - \Delta C_2(S \times \bar{F} \to F) \\ = \ & 2C_2([N/2]_N) - C_2(S) = \frac{2k^3 - 3k^2 - 2k + 2}{2k} \ . \end{aligned} \tag{4.54}$$

This difference is positive for all values of $k \geq 2$ of interest here.

If the beta function has no IR zero, then as the scale μ decreases and $\alpha(\mu)$ increases, it will eventually become large enough to cause condensation, which, according to the MAC criterion, will be in this channel (4.51). If the beta function does have a zero, then the next step in the analysis is to determine how the value of the coupling at this zero compares with α_{cr} for the most attractive channel, (4.51). Substituting (4.52) into the general formula for Eq. (3.4), we calculate

$$\alpha_{cr,AT} = \frac{4\pi}{3k(2k+1)} \ . \tag{4.55}$$

As discussed above, an approximate measure of how strong the coupling gets in the infrared, compared with the minimum critical value for condensation in the MAC is then given by the ratio

$$\rho_{_{IR,AT}} \equiv \frac{\alpha_{_{IR,2\ell,AT}}}{\alpha_{cr,AT}} \ . \tag{4.56}$$

We list values of $\rho_{_{IR,AT}}$ for the relevant N and p in Table 4.4. In cases where condensation occurs in this theory we denote the scale at which it occurs as $\Lambda_{[N/2]_N}$.

AT Theory with $G = \mathrm{SU}(4)$

In this subsection and the following ones we discuss three illustrative cases with various values of $N = 2k$ and their corresponding intervals $(I_p)_{AT}$. For each value of N, if p is nonzero and $p < p_{b2z}$, i.e., below the lower end of the interval $(I_p)_{AT}$, then the theory has no IR fixed point, even an approximate one, so that the gauge coupling continues to grow in the infrared and will cause condensation in the MAC. Hence, we restrict our consideration here to $p \in (I_p)_{AT}$. The reader is referred to Tables 4.5 and 4.4 for numerical values of relevant quantities. As indicated in Table 4.5, for this SU(4) AT theory the interval $(I_p)_{AT}$ is $3 \le p \le 14$. For p in this interval, $\beta_{\alpha,2\ell,AT}$ has an IR zero at

$$N = 4: \ \alpha_{IR,2\ell,AT} = \frac{8\pi(15-p)}{55p-138} \ . \tag{4.57}$$

The ratio $\rho_{IR,AT}$ is

$$N = 4: \ \rho_{IR,AT} = \frac{60(15-p)}{55p-138} \ . \tag{4.58}$$

As listed in Table 4.4, for the range of p from 3 to 7, this ratio takes on values decreasing from 26.7 to 1.94, all well above unity. Thus, one may plausibly expect that for these values of p, in the UV to IR evolution, as the reference scale μ decreases sufficiently and the running coupling approaches $\alpha_{IR,2\ell,AT}$, the gauge interaction will become strong enough to cause fermion condensation in the most attractive channel, $[2]_4 \times [2]_4 \to 1$. For $p = 8, \ 9, \ 10, \ 11$, $\rho_{IR,AT}$ has the respective values $1.39, \ \ 1.01, \ \ 0.728, 0.514$. Given the theoretical uncertainties in these estimates, the IR behavior might or might not involve the formation of the condensates (4.53). For the largest values of p, namely $p = 12, \ 13, \ 14$, $\rho_{IR,AT}$ has the respective values $0.345, \ \ 0.208, \ \ 0.095$, so for these cases, it is likely that the theory evolves from the UV to a scale-invariant, deconfined, Coulombic IR phase. This inference is, of course, most reliable for the largest allowed value of p, namely $p = 14$, which leads to the smallest value of $\alpha_{IR,2\ell,AT}$ and $\rho_{IR,AT}$. As discussed above, in the cases where there is condensate formation and chiral symmetry breaking, the IRFP is only approximate, while in the cases where there is no such chiral symmetry breaking the IRFP (calculated to all orders) is exact.

Table 4.5: Values of $p_{b1z,AT}$, $p_{b2z,AT}$, p_{max}, and the intervals $(I_p)_{AT}$ as functions of N in the AT model with gauge group SU(N) with $N = 2k$.

N	$p_{b2z,AT}$	$p_{b1z,AT}$	p_{max}	$(I_p)_{AT}$
4	2.509	15	14	$3 \le p \le 14$
6	1.590	8	7	$2 \le p \le 7$
8	0.665	3.3	3	$1 \le p \le 3$
10	0.235	1.2	1	$p = 1$

AT Theory with $G = \mathrm{SU}(6)$

In the SU(6) (i.e., $k = 3$) AT theory, $(I_p)_{AT}$ is the interval $2 \le p \le 7$. For p in this interval, $\beta_{\alpha,2\ell}$ has an IR zero at

$$N = 6: \ \alpha_{IR,2\ell,AT} = \frac{16\pi(8-p)}{3(61p-97)} \ . \tag{4.59}$$

The ratio $\rho_{IR,AT}$ is

$$N = 6: \ \rho_{IR,AT} = \frac{84(8-p)}{61p-97} \ . \tag{4.60}$$

As listed in Table 4.4, for $2 \le p \le 7$, this has the respective values 20.16, 4.89, 2.29, 1.21, 0.625, 0.255. Thus, for $p = 3$ and $p = 4$, it is likely that condensation occurs in the MAC, $[3]_6 \times [3]_6 \to 1$ channel; for $p = 7$, it is likely that there is no condensation; and for the middle two values $p = 5$ and $p = 6$, taking account of the intrinsic theoretical uncertainties, one cannot give a very definite prediction from this analysis.

AT Theory with $G = \mathrm{SU}(10)$

In the SU(10) ($k = 5$) AT theory, the interval $(I_p)_{AT}$ reduces to just a single nonzero value, $p = 1$, and the resultant $\alpha_{IR,2\ell,AT} = 0.036$, yielding the ratio $\rho_{IR,AT} = 0.473$. It is thus likely that this theory evolves from the UV to the IR to a non-Abelian Coulomb phase, although there are obvious uncertainties in this inference due to the strong-coupling physics involved.

4.4 Global Flavor Symmetry for Theories with Self-Conjugate R

In analyzing the global flavor symmetry of these chiral gauge theories, it is useful to consider a more general class of theories, in which the vectorlike fermion subsector is comprised of fermions transforming under a general self-conjugate representation, $R = R_{sc}$. The results will then be applied to the two specific theories discussed above, namely those with $G = \mathrm{SU}(N)$, $N \ge 3$, and $R_{sc} = Adj$; and the AT theory with $G = \mathrm{SU}(N)$ with even $N = 2k$, $k \ge 2$, and $R_{sc} = [N/2]_N$.

The classical global chiral flavor symmetry of a theory in this class of theories is

$$\begin{aligned} G_{fl,cl,R_{sc}} &= \mathrm{U}(1)_S \otimes \mathrm{U}(N+4)_{\bar{F}} \otimes \mathrm{U}(p)_{R_{sc}} \\ &= \mathrm{U}(1)_S \otimes \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{U}(1)_{\bar{F}} \otimes \mathrm{SU}(p)_{R_{sc}} \otimes \mathrm{U}(1)_{R_{sc}} \ . \end{aligned} \tag{4.61}$$

The representations of the fermions in the two theories with $R = R_{sc}$ under this symmetry are given in Table 4.1. The corresponding global unitary transformations are

$$\psi_L^{ab} \to U_S\, \psi_L^{ab} \ , \quad U_S \in \mathrm{U}(1)_S \ , \tag{4.62}$$

$$\chi_{a,i,L} \to \sum_{j=1}^{N+4} (U_{\bar{F}})_{ij}\, \chi_{a,j,L} \ , \quad U_{\bar{F}} \in \mathrm{U}(N+4)_{\bar{F}} \ , \tag{4.63}$$

and

$$\xi_{i,L} \to \sum_{j=1}^{p} (U_{R_{sc}})_{ij}\, \xi_{j,L} \ , \quad U_{R_{sc}} \in \mathrm{U}(p)_{R_{sc}} \tag{4.64}$$

where we have suppressed the SU(N) gauge indices in Eq. (4.64), which applies to each theory with the corresponding ξ field, i.e., $\xi^a_{b,i,L}$ in the Adj theory and $\xi^{a_1,\ldots,a_k}_{i,L}$ in the AT theory.

Each of the three global U(1) symmetries is broken the instantons of the SU(N) gauge theory [103]. One may define a three-dimensional vector of anomaly factors,

$$\vec{v} \ = \ \Big(N_S T(S),\ N_{\bar{F}} T(\bar{F}),\ N_{R_{sc}} T(R_{sc}) \Big)$$

$$= \left(\frac{N+2}{2},\ \frac{N+4}{2},\ pT_{R_{sc}} \right) , \tag{4.65}$$

where the basis is $(S, \bar{F}, R_{sc})$, and we have inserted the values $N_S = 1$, $N_{\bar{F}} = N + 4$, and $N_{R_{sc}} = p$. One can construct two linear combinations of the three original currents that are conserved in the presence of SU(N) instantons. The fermions have charges under these global U(1) symmetries given by the vectors

$$\vec{Q}^{(j)} \equiv \left(Q_S^{(j)},\ Q_{\bar{F}}^{(j)}, Q_{R_{sc}}^{(j)} \right), \quad j = 1, 2 , \tag{4.66}$$

where $j = 1$ for $U(1)_1$ and $j = 2$ for $U(1)_2$. The condition that the corresponding currents are conserved, i.e., the $U(1)_j$ global symmetries are exact, in the presence of instantons is that

$$\sum_f N_f T(R_f)\, Q_f^{(j)} = \vec{v} \cdot \vec{Q}^{(j)} = 0 \quad \text{for } j = 1,\ 2 . \tag{4.67}$$

As indicated, this condition is equivalent to the condition that the vectors of charges under the $U(1)_1$ and $U(1)_2$ symmetries are orthogonal to the vector $\vec{v}$. (Note that the condition (4.67) does not uniquely determine the vectors $\vec{Q}^{(j)}$, $j = 1,\ 2$. It will be convenient to choose the first vector, $\vec{Q}^{(1)}$, so that $Q_{R_{sc}}^{(1)} = 0$. We thus choose

$$\vec{Q}^{(1)} = \Big(N+4,\ -(N+2),\ 0 \Big) . \tag{4.68}$$

For the vector of charges under $U(1)_2$, we choose

$$\vec{Q}^{(2)} = \Big(2pT_{R_{sc}},\ 0,\ -(N+2) \Big) . \tag{4.69}$$

(Note that in contrast to Gram-Schmidt orthogonalization of the three vectors $\vec{v}$, $\vec{Q}^{(1)}$, and $\vec{Q}^{(2)}$, here it is not necessary that $\vec{Q}^{(1)} \cdot \vec{Q}^{(2)} = 0$.)

The actual non-anomalous global chiral flavor symmetry group of the class of chiral gauge theories with $R = R_{sc}$ is then

$$G_{fl,R_{sc}} = \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{SU}(p)_{R_{sc}} \otimes \mathrm{U}(1)_1 \otimes \mathrm{U}(1)_2 . \tag{4.70}$$

For the two respective theories with (i) $R_{sc} = Adj$ and (ii) $R_{sc} = [N/2]_N$, Eqs. (4.69) and (4.70) apply with (i) $T_{R_{sc}} = T(Adj) = N$ and (ii) $T_{[N/2]_N}$ given by Eq. (A.11) in Appendix A. We summarize these properties in Table 4.1.

In general, one must also check to see if either of the chiral gauge theories with $R_{sc} = Adj$ or $R_{sc} = [N/2]_N$ satisfies the 't Hooft anomaly-matching conditions, which are necessary conditions for the possible formation of massless gauge-singlet composite fermions. The possible gauge-singlet fermions can be described by wavefunctions of the form

$$B_{ij} = \bar{F}_{a,i,L}\, S_L^{ab}\, \bar{F}_{b,j,L} \ , 1 \le i,j \le N+4 \ . \tag{4.71}$$

Given the minus sign from Fermi statistics and the fact that S^{ab} is a rank-2 symmetric tensor representation (▭▭) of SU(N), it follows that $B_{ij} = -B_{ji}$, i.e., B_{jk} is a rank-2 antisymmetric tensor representation (⊟) of the SU$(N+4)_{\bar{F}}$ factor group in the global flavor symmetry group G_{fl}. There are thus $(N+4)(N+3)/2$ components of B_{ij}. The charges of B_{ij} under the two global abelian factor groups in $G_{fl,R_{sc}}$, U$(1)_k$, $k = 1,2$ are determined by the relation

$$Q_B^{(k)} = Q_S^{(j)} + 2Q_{\bar{F}}^{(k)} \ , \quad k = 1,2 \tag{4.72}$$

Hence,

$$Q_B^{(1)} = -N \tag{4.73}$$

and

$$Q_B^{(2)} = 2p\, T_{R_{sc}} \ . \tag{4.74}$$

We find that the global anomalies of a theory with these massless composite fermions do not match those of the original G_{fl} group except in the degenerate case $p = 0$. This $p = 0$ case describes a descendant low-energy effective field theory that occurs if there is condensation in the $R_{sc} \times R_{sc} \to 1$ channel, and will be discussed below.

4.5 Analysis of Low-Energy Effective Theory for $\mu < \Lambda_{R_{sc}}$

In the cases where the values of N and p are such as to lead to the respective bilinear fermion condensates (4.29) or (4.53) at the corresponding scales Λ_{Adj} or $\Lambda_{[N/2]_N}$, we analyze the further UV to IR evolution below these scales. We denote these scales generically as $\Lambda_{R_{sc}}$. Because of this condensation, the p fermions $\xi^a_{b,i,L}$ involved in the condensate (4.29) in the *Adj* model and the p fermions $\xi^{a_1,\ldots,a_l}_{i,L}$ involved in the condensate (4.53) in the AT theory gain dynamical masses of order Λ_{Adj} and $\Lambda_{[N/2]_N}$, respectively.

For momentum scales μ slightly below the condensation scale $\Lambda_{R_{sc}}$, the resultant global symmetry is

$$G'_{fl} = \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{SO}(p) \otimes \mathrm{U}(1)_1 \ . \tag{4.75}$$

Here the $\mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{U}(1)_1$ is a global chiral symmetry operating on the massless S and $\bar{F}$ fermions, leaving their covariant derivatives invariant, while the $\mathrm{SO}(p)$ is a global isospin symmetry of the condensate in each of our two theories with $R = R_{sc}$, or equivalently, the corresponding effective mass term. These mass terms are

$$\Lambda_{Adj} \sum_{i=1}^{p} \xi^{a\ T}_{b,i,L} C\, \xi^b_{a,i,L} + h.c. \tag{4.76}$$

in the *Adj* theory and

$$\Lambda_{[N/2]_N} \sum_{i=1}^{p} \langle \epsilon_{a_1,\ldots a_{2k}}\, \xi^{a_1,\ldots,a_k\ T}_{i,L} C\, \xi^{a_{k+1},\ldots,a_{2k}}_{i,L} \rangle + h.c. \tag{4.77}$$

in the AT theory produced by the bilinear fermion condensations in these respective theories. This $\mathrm{SO}(p)$ symmetry also leaves the covariant derivatives of these ξ fields invariant.

The spontaneous symmetry breaking of the initial nonanomalous global symmetry G_{fl} in Eq. (4.70) to the final global symmetry Eq. (4.75) produces

$$o(\mathrm{SU}(p)) + 1 - o(\mathrm{SO}(p)) = \frac{p(p+1)}{2} \ . \tag{4.78}$$

massless Nambu-Goldstone bosons, where $o(H)$ denotes the order of a group H.

As the reference scale μ decreases well below $\Lambda_{R_{sc}}$, we integrate these now-massive ξ fermions out of the low-energy (LE) effective field theory (LEEFT) applicable for $\mu \ll \Lambda_{R_{sc}}$. Focusing on the infrared region $\mu \ll \Lambda_{R_{sc}}$, with the ξ fermions integrated out, both the theory with $R_{sc} = Adj$ and the theory with $R_{sc} = [N/2]_N$ reduce to the same low-energy descendant theory, with (massless) S fermion and $N+4$ copies of $\bar{F}$ fermions. We denote this as the $S\bar{F}$ theory. This theory has been well studied in the past [9,11,18,22,82,83]. We recall the results from these earlier studies that we will need for our present analysis.

The value of f_{UV} for the $S\bar{F}$ model, which we denote as $f_{UV,S\bar{F}M}$ (M standing for model), is given by the $p=0$ special case of Eq. (4.89), namely

$$f_{UV,S\bar{F}M} = 2(N^2-1) + \frac{7}{4}\Big[\frac{N(N+1)}{2} + (N+4)N\Big] \ . \qquad (4.79)$$

The $S\bar{F}$ theory is invariant under a nonanomalous global flavor symmetry group

$$G_{fl,S\bar{F}M} = \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{U}(1)_{S\bar{F}} \ . \qquad (4.80)$$

For this theory the three-dimensional vector (4.65) reduces to a two-dimensional vector with the third entry deleted, and the vector of charges that is orthogonal to it and hence defines the charge assignments of the $\mathrm{U}(1)_{S\bar{F}}$ is given by the first two entries in $Q^{(1)}$, namely

$$\vec{Q}^{(1)} = (N+4, -(N+2)) \ . \qquad (4.81)$$

The $S\bar{F}$ theory is asymptotically free, so the gauge coupling continues to grow as μ decreases. The beta function of this $S\bar{F}$ theory has one-loop and two-loop coefficients given by Eqs. (4.8) and (4.9) with $p=0$. In the relevant range $N \ge 3$, b_2 is positive. Since b_1 and b_2 thus have the same sign, the beta function, calculated to the maximal scheme-independent order of two loops, does not have any IR zero. Hence, as μ decreases from the UV to the IR, the running coupling $\alpha(\mu)$ increases, eventually exceeding the region where the perturbatively calculated beta function is applicable.

There are two possible types of UV to IR evolution in the $S\bar{F}$ theory. First, the strongly coupled gauge interaction may produce bilinear fermion condensates. The most attractive channel is $S \times \bar{F} \to F$, with condensates

$$\langle \sum_{b=1}^{N} \psi_L^{ab\ T} C \chi_{b,i,L} \rangle \ . \qquad (4.82)$$

Without loss of generality, one may take $a=N$ and $i=1$ for the first condensate. This breaks the $\mathrm{SU}(N)$ gauge symmetry down to $\mathrm{SU}(N-1)$, so that the $2N-1$ gauge bosons in the coset $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ gain masses of order this scale of condensation, which we denote Λ_N. The fermions ψ_L^{Nb}

and $\chi_{b,1,L}$ with $b = 1, .., N$ involved in this condensate also gain dynamical masses of order Λ_N. In the low-energy theory applicable for scales $\mu < \Lambda_N$, these now massive fermions are integrated out.

The descendant theory is again asymptotically free, so the gauge coupling inherited from the SU(N) theory continues to increase. There is then a second condensation, again in the MAC, $S \times \bar{F} \to F$ channel, breaking the gauge symmetry from SU($N-1$) to SU($N-2$). Without loss of generality, we may take the breaking direction to be $a = N-1$ and the $\bar{F}$ fermion involved in the condensate to be labelled as $\chi_{b,2,L}$, so that this condensate is

$$\langle \sum_{b=1}^{N-1} \psi_L^{N-1,b\ T} C \chi_{b,2,L} \rangle \ . \tag{4.83}$$

We denote the scale at which this occurs as Λ_{N-1}. The $2N-3$ gauge bosons in the coset SU($N-1$)/SU($N-2$) gain masses of order Λ_{N-1} and the fermions $S_L^{N-1,b}$ and $\chi_{b,2,L}$ with $b = 1, ..., N-1$ involved in this condensate gain dynamical masses of order Λ_{N-1}. This sequential breaking via condensation in the respective $S \times \bar{F} \to F$ channels continues at the scales Λ_{N-2}, etc. until the gauge symmetry is completely broken. Thus, the sequence of gauge symmetry breaking is

$$\mathrm{SU}(N) \to \mathrm{SU}(N-1) \to \cdots \to SU(2) \to \emptyset \ . \tag{4.84}$$

The gauge bosons in the respective cosets SU(N)/SU($N-1$), SU($N-1$)/SU($N-2$), etc. gain masses of order Λ_N, Λ_{N-1}, etc, as do the components of the fermions involved in the respective condensates.

Considering the $S\bar{F}$ theory, for this type of UV to IR evolution [22,82,83],

$$f_{IR,S\bar{F}M;S\times\bar{F}} = 8N + 1 + \frac{7}{4}\Big[\frac{N(N-1)}{2} + 4N\Big] \ , \tag{4.85}$$

where the subscript $S\bar{F}M$ means the $S\bar{F}$ model, and the subscript $S\bar{F}$ refers to the condensation channel. For the $S\bar{F}$ model, with this type of UV to IR evolution, one then has

$$(\Delta f)_{S\bar{F}M;S\times\bar{F}} = f_{UV,S\bar{F}M} - f_{IR,S\bar{F}M;S\times\bar{F}} = \frac{15N^2 - 25N - 12}{4} \ . \tag{4.86}$$

This is positive for all relevant values of N. (For N extended to the positive reals, it is positive for $N > (25 + \sqrt{1345}\ )/30 = 2.056$.)

The low-energy effective $S\bar{F}$ theory applicable below $\Lambda_{R_{sc}}$ could also undergo a different type of flow deeper into the infrared, namely one leading to confinement with massless gauge-singlet composite fermions with wavefunctions (4.71). In this case, for this $S\bar{F}$ theory, considered in isolation,

$$f_{IR,S\bar{F}M;sym} = \frac{7}{4}\Big[\frac{(N+4)(N+3)}{2}\Big] \ . \tag{4.87}$$

Hence, for this type of UV to IR evolution,

$$(\Delta f)_{S\bar{F}M;sym} = \frac{15N^2+7N-50}{4} \ . \tag{4.88}$$

This is positive for all relevant values of N. (For N extended to the positive reals, it is positive for $N > (-7+\sqrt{3049}\ )/30 = 1.607$.) Thus, for both of these types of UV to IR evolution of the $S\bar{F}$ theory, the conjectured degree-of-freedom inequality (3.12) is obeyed.

4.6 Comparison with Degree-of-Freedom Inequality

We now combine the results for the $S\bar{F}$ theory with our calculations of UV and IR degree-of-freedom counts for the different types of UV to IR evolution in the *Adj* and AT chiral gauge theories and compare with the conjectured degree-of-freedom inequality Eq. (3.12).

4.6.1 UV Count

Given that we have required our theories to be asymptotically free, they are weakly coupled in the UV, so we can identify the perturbative degrees of freedom and calculate f_{UV}. From the general formula Eq. (3.10), we have

$$f_{UV,R_{sc}} = 2(N^2-1) + \frac{7}{4}\Big[\frac{N(N+1)}{2} + (N+4)N\Big] + \frac{7}{8}\,p\,\dim(R_{sc}) \ , \tag{4.89}$$

where the respective terms represent the contributions of the SU(N) gauge fields, the S fermions, the $N+4$ copies of $\bar{F}$ fermions, and the R_{sc} fermions. Explicitly, for the *Adj* theory,

$$f_{UV,Adj} = 2(N^2-1) + \frac{7}{4}\Big[\frac{N(N+1)}{2} + (N+4)N\Big] + \frac{7}{8}\,p\,(N^2-1) \tag{4.90}$$

and for the AT theory, with $N = 2k$,

$$f_{UV,AT} = 2(N^2-1) + \frac{7}{4}\Big[\frac{N(N+1)}{2} + (N+4)N\Big] + \frac{7}{8}\, p \begin{pmatrix} N \\ N/2 \end{pmatrix} . \quad (4.91)$$

where $\binom{a}{b}$ is the binomial coefficient.

4.6.2 f_{IR} Calculations

Next, we calculate f_{IR} for the two types of chiral gauge theories discussed above in the cases where the UV to IR evolution involves a high-scale condensation in the respective channels (4.26) or (4.51), followed by sequential condensations in the $S \times \bar{F} \to F$ channel. Taking account of the $p(p+1)/2$ NGBs from the higher-scale symmetry breaking at $\Lambda_{R_{sc}}$, we find, for either of these two types of chiral gauge theories, for this type of infrared evolution below $\Lambda_{R_{sc}}$,

$$\begin{aligned} & f_{IR,Adj;Adj\times Adj,S\times\bar{F}} = f_{IR,AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} \equiv f_{IR,R_{sc};R_{sc}\times R_{sc},S\times\bar{F}} \\ = \; & 8N+1+\frac{7}{4}\Big[\frac{N(N-1)}{2}+4N\Big]+\frac{p(p+1)}{2} \ , \end{aligned} \quad (4.92)$$

where the subscript R_{sc} identifies the chiral fermion representation in the vectorlike subsector, the next subscript $R_{sc} \times R_{sc}$ is shorthand for the MAC $R_{sc} \times R_{sc} \to 1$ in which the highest-scale condensation takes place, and the last subscript, $S \times \bar{F}$ or *sym* are shorthand for the two types of IR flow in the low-energy descendant theory, namely sequential $S \times \bar{F} \to F$ condensation formation and gauge and global symmetry breaking in the descendent theory, or confinement with formation of massless composite fermions and retention of exact chiral symmetry (*sym*) in the infrared. Thus, the subscripts here and below placed after the semicolon in quantities such as $f_{IR,Adj;Adj\times Adj,S\times\bar{F}}$ refer to the sequence of steps in the UV to IR evolution.

For the alternate type of evolution involving high-scale condensation in the respective channels (4.26) or (4.51), followed by confinement leading to massless gauge-singlet composite fermions, we calculate, for either of our two types of chiral gauge theory with $R = R_{sc}$,

$$\begin{aligned} & f_{IR,Adj;Adj\times Adj,sym} = f_{IR,AT;[k]_{2k}\times[k]_{2k},sym} \equiv f_{IR,R_{sc};R_{sc}\times R_{sc},sym} \\ = \; & \frac{7}{4}\Big[\frac{(N+4)(N+3)}{2}\Big]+\frac{p(p+1)}{2} \ . \end{aligned} \quad (4.93)$$

4.6.3 Comparison with DFI for *Adj* Theory

Using these inputs, we can now calculate Δf for these chiral gauge theories and compare with the conjectured degree-of-freedom inequality (3.12). For both theories, if the UV to IR evolution is such as to lead to a deconfined non-Abelian Coulomb phase, the perturbative degrees of freedom are the same as in the UV, so the DFI is obeyed. (The perturbative corrections also obey the DFI [83, 84].)

We first discuss the possible cases for the theory with $R = Adj$. If N and p are such that the gauge interaction produces the high-scale condensation in the channel (4.29), followed by Eqs. (4.78) with (4.85), we calculate

$$(\Delta f)_{Adj;Adj\times Adj,S\times\bar{F}} \equiv f_{UV,Adj} - f_{IR,Adj;Adj\times Adj,S\times\bar{F}}$$
$$= \frac{1}{8}\Big[30N^2 - 50N - 24 + 7pN^2 - 11p - 4p^2\Big] \ . \qquad (4.94)$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\Big[7N^2 - 11 + \sqrt{49N^4 + 326N^2 - 800N - 263}\ \Big] \ . \qquad (4.95)$$

The upper bound on the right-hand side of Eq. (4.95) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (4.11). Hence, the conjectured degree-of-freedom inequality (3.12) is obeyed for all N and allowed p with this type of UV to IR evolution.

For the case where the low-energy effective $S\bar{F}$ theory confines without any spontaneous chiral symmetry breaking, producing massless composite fermions, we calculate

$$(\Delta f)_{Adj;Adj\times Adj,sym} \equiv f_{Adj,UV} - f_{IR,Adj;Adj\times Adj,sym}$$
$$= \frac{1}{8}\Big[30N^2 + 14N - 100 + 7pN^2 - 11p - 4p^2\Big] \ . \qquad (4.96)$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\Big[7N^2 - 11 + \sqrt{49N^4 + 326N^2 + 224N - 1479}\ \Big] \ . \qquad (4.97)$$

The upper bound on the right-hand side of Eq. (4.97) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (4.11).

Hence, the conjectured degree-of-freedom inequality (3.12) is also obeyed for all N and allowed p with this type of UV to IR evolution.

As illustrative numerical examples, we may consider the cases $N = 3$ and $N = 4$. In these cases, the respective upper bounds on p from Eq. ((4.11) are $p \le 3$, while the respective values of the right-hand side of (4.95) are 14.64 and 27.57 and the respective values of the right-hand side of (4.97) are 16.26 and 29.01. Note that if p is close to the upper bound p_{b1z} arising from the requirement of asymptotic freedom, then b_1 is small, so that $\alpha_{IR,2\ell}$ is sufficiently small that the UV to IR evolution is to a non-Abelian Coulomb phase, so that one knows that the DFI is satisfied without going through the present analysis.

These expressions simplify in the limit $N \to \infty$ (with p fixed) in Eq. (4.18). We define rescaled degree-of-freedom measures that are finite in this limit, of the form

$$\bar{f} \equiv \lim_{N\to\infty} \frac{f}{N^2} \ . \tag{4.98}$$

(We use the same notation, $\bar{f}$ for this $N \to \infty$ limit; the context will always make clear which limit is meant.) We calculate

$$\bar{f}_{UV,Adj} = \frac{37+7p}{8} \ , \tag{4.99}$$

$$\bar{f}_{IR,Adj;Adj\times Adj,S\times\bar{F}} = \bar{f}_{IR,Adj;Adj\times Adj,sym} = \frac{7}{8} \ , \tag{4.100}$$

and hence

$$(\Delta\bar{f})_{Adj;Adj\times Adj,S\times\bar{F}} = (\Delta\bar{f})_{Adj;Adj\times Adj,sym} = \frac{30+7p}{8} \ . \tag{4.101}$$

This obviously obeys the conjectured degree-of-freedom inequality (3.12).

4.6.4 Comparison with DFI for AT Theory

We next calculate Δf for the AT chiral gauge theory with gauge group $G = {\rm SU}(N)$ with even $N = 2k$ and $R_{sc} = [N/2]_N = [k]_{2k}$. As noted above, for values of N and p such that the UV to IR evolution is to a deconfined non-Abelian Coulomb phase in the IR, the perturbative degrees of freedom are the same as in the UV, and the conjectured degree-of-freedom inequality is obeyed.

If N and p are such that the gauge interaction produces high-scale condensation in the channel (4.26) followed at lower scales by condensations in the successive $S \times \bar{F} \to F$ channels in SU(N), SU($N-1$), etc., then, using Eqs. (4.78) and (4.85), we compute

$$\begin{aligned}(\Delta f)_{AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} &\equiv f_{UV,AT} - f_{IR,AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} \\ &= \frac{1}{8}\Big[30N^2 - 50N - 24 + 7p\,d_R - 4p(p+1)\Big] \ , \end{aligned} \tag{4.102}$$

where $d_R \equiv \binom{N}{N/2}$. This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\Big[7d_R - 4 + \sqrt{480N^2 - 800N - 368 + 49d_R^2 - 56d_R}\ \Big] \ . \tag{4.103}$$

The upper bound on the right-hand side of Eq. (4.103) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (4.46). Hence, the conjectured degree-of-freedom inequality Eq. (3.12) is also obeyed for all N and allowed p with this type of UV to IR evolution in the AT model.

For the alternate type of UV to IR evolution in which the low-energy effective $S\bar{F}$ theory confines without any spontaneous chiral symmetry breaking, producing massless composite fermions, we calculate

$$\begin{aligned}(\Delta f)_{AT;[k]_{2k}\times[k]_{2k},sym} &\equiv f_{AT,UV} - f_{IR,AT;[k]_{2k}\times[k]_{2k},sym} \\ &= \frac{1}{8}\Big[30N^2 + 14N - 100 + 7p\,d_R - 4p(p+1)\Big] \ . \end{aligned} \tag{4.104}$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\Big[7d_R - 4 + \sqrt{480N^2 + 224N - 1584 + 49d_R^2 - 56d_R}\ \Big] \ . \tag{4.105}$$

The upper bound on the right-hand side of Eq. (4.105) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (4.46). Hence, the conjectured degree-of-freedom inequality Eq. (3.12) is also obeyed for all N and allowed p with this type of UV to IR evolution in the AT model.

As numerical examples, for $N = 4$ and $N = 6$, the respective upper bounds on p from Eq. (4.46) are $p \le 14$ and $p \le 7$, while the respective right-hand sides of (4.103) are 14.05 and 38.86 and the respective right-hand sides of (4.105) are 16.22 and 40.56. As before, it should be noted that if p is close to the upper bound from asymptotic freedom, b_1 is small, so that $\alpha_{IR,2\ell}$ is sufficiently small that the UV to IR evolution is to a non-Abelian Coulomb phase, so that one knows that the conjecture degree-of-freedom inequality Eq.(3.12) is satisfied.

Chapter 5

Chiral Gauge Theories with Fermions in Symmetric and Antisymmetric Rank-2 Representations

In this chapter, we study the ultraviolet to infrared evolution and nonperturbative behavior of a simple set of asymptotically free chiral gauge theories with an SU(N) gauge group and an anomaly-free set of n_{S_k} copies of chiral fermions transforming as the symmetric rank-k tensor representation, S_k, and $n_{\bar{A}_\ell}$ copies of fermions transforming according to the conjugate antisymmetric rank-ℓ tensor representation, $\bar{A}_\ell$, of this group with $k, \ \ell \geq 2$. We analyze the theories with $k = \ell = 2$ in detail and show that there are only a finite number of these. Depending on the specific theory, the ultraviolet to infrared evolution may lead to a non-Abelian Coulomb phase, or may involve confinement with massless composite fermions, or fermion condensation with dynamical gauge and global symmetry breaking. We show that $S_k\bar{A}_k$ chiral gauge theories with $k \geq 3$ are not asymptotically free. We also analyze theories with fermions in S_k and $\bar{A}_\ell$ representations of SU(N) with $k \neq \ell$ and $k, \ \ell \geq 2$. The result presented in this chapter is based on publication [2].

5.1 $S\bar{A}$ Theories

5.1.1 Basic Construction

In this section we construct and analyze an interesting set of asymptotically free chiral gauge theories with an SU(N) gauge group and chiral fermions transforming according to the rank-2 symmetric and conjugate antisymmetric tensor representations of this group. We denote these fermions generically as S_2, and $\bar{A}_2$ (suppressing possible flavor indices) and equivalently by the corresponding Young tableaux, $S_2 = \square\square$ and $\bar{A}_2 = \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$. To keep the notation as simple as possible, we omit the subscripts where no confusion will result, setting

$$S_2 \equiv S, \quad \bar{A}_2 \equiv \bar{A} \ . \tag{5.1}$$

We denote the number of S and $\bar{A}$ fields as n_S and $n_{\bar{A}}$. An $S\bar{A}$ theory is irreducibly chiral, i.e., it does not contain any vectorial subset. The chiral gauge symmetry forbids any fermion mass term in the Lagrangian. The triangle anomaly $\mathcal{A}$ in gauged currents of our $S\bar{A}$ theory is

$$\begin{aligned}\mathcal{A} &= n_S\mathcal{A}(S) + n_{\bar{A}}\mathcal{A}(\bar{A}) \\ &= n_S\mathcal{A}(S) - n_{\bar{A}}\mathcal{A}(A) \ .\end{aligned} \tag{5.2}$$

Substituting $\mathcal{A}(S) = N + 4$ and $\mathcal{A}(A) = N - 4$ (see Appendix A), the condition that this $S\bar{A}$ theory should be free of a triangle anomaly in gauged currents is that

$$n_S(N+4) - n_{\bar{A}}(N-4) = 0 \ . \tag{5.3}$$

Thus, n_S and $n_{\bar{A}}$ take values in the ranges $n_S \geq 1$ and $n_{\bar{A}} \geq 1$, subject to the anomaly cancellation condition (5.3) and the requirement that the resultant $S\bar{A}$ theory must be asymptotically free. A member of this set of chiral gauge theories is thus defined as

$$S\bar{A}: \quad G = \mathrm{SU}(N), \quad \text{fermions}: \ n_S S + n_{\bar{A}}\bar{A} \ , \tag{5.4}$$

where it is understood implicitly that N, n_S, and $n_{\bar{A}}$ satisfy the condition (5.3) and yield an asymptotically free theory. We denote such a theory, for short, as $(N; n_S, n_{\bar{A}})$.

The anomaly cancellation condition (5.3) is a linear diophantine equation. If $N = 4$, i.e., $G = \mathrm{SU}(4)$, a nontrivial solution of this equation is not possible, because in this case the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation is self-conjugate, and hence has zero anomaly, so there is no value of $n_{\bar{A}}$ which can cancel the contribution to the anomaly in gauged currents from the fermions in the $\square\square$ representation. Consequently, a nontrivial solution of the anomaly cancellation condition (5.3) requires that $N \geq 5$, and we restrict to this range. We define the ratio

$$\frac{n_{\bar{A}}}{n_S} = \frac{N+4}{N-4} \equiv p \ . \tag{5.5}$$

Since the right-hand side of Eq. (5.5) is greater than one, it follows that $n_S < n_{\bar{A}}$. Therefore, the theories of this type with minimal chiral fermion content have $n_S = 1$ and take the form

$$(N; n_S, n_{\bar{A}}) = (N; 1, p) \ , \tag{5.6}$$

with the understanding that $n_{\bar{A}}$ must be a (positive) integer. We find that there are precisely four solutions of Eq. (5.5) with $n_S = 1$ that satisfy this condition, namely (including the N characterizing the SU(N) gauge group)

$$(N; n_S, n_{\bar{A}}) = (5; 1, 9), \ (6; 1, 5), \ (8; 1, 3), \ (12; 1, 2) \ . \tag{5.7}$$

In the context of anomaly cancellation alone, before imposing the condition of asymptotic freedom, we observe a basic mathematical property. If $(N; n_S, n_{\bar{A}})$ is a solution of Eq. (5.3), then a theory with n_{cp} copies (abbreviated *cp*) of the fermion content also yields a solution of (5.3). That is,

$$\begin{aligned} &(N; n_S, n_{\bar{A}}) \text{ is anom. free } \Longrightarrow \\ &(N; n_{cp} n_S, n_{cp} n_{\bar{A}}) \text{ is anom. free for } n_{cp} \geq 2. \end{aligned} \tag{5.8}$$

If one were not to require that the theory must be asymptotically free, then n_{cp} could be any positive integer, and hence the linear diophantine equation (5.3) would have an infinite number of solutions. However, we do require that our chiral gauge theories must be asymptotically free so that they are perturbatively calculable in the deep ultraviolet.

Given this requirement, the next step is to ascertain, for a given value of N, which values of n_{cp} are allowed by asymptotic freedom. To do this, we calculate the first two coefficients of the beta function. These coefficients are

$$b_1 = \frac{1}{3}\left[11N - \left\{ n_S(N+2) + n_{\bar{A}}(N-2) \right\}\right] \tag{5.9}$$

and

$$
\begin{aligned}
b_2 &= \frac{1}{3}\Bigg[34N^2 - n_S\Bigg\{5N + 3\frac{(N+2)(N-1)}{N}\Bigg\}(N+2) \\
&\quad -n_{\bar{A}}\Bigg\{5N + 3\frac{(N-2)(N+1)}{N}\Bigg\}(N-2)\Bigg] \ . \qquad (5.10)
\end{aligned}
$$

It will be useful to reexpress these coefficients in a convenient form for analysis of the minimal set of fermions, viz., $(n_S, n_{\bar{A}}) = (1, p)$, given explicitly in (5.7) and the sets involving n_{cp}-fold replication (copies, or flavors) of minimal sets,

$$
(n_S, n_{\bar{A}}) = n_{cp}(1, p) = (n_{cp}, n_{cp}p) \ . \qquad (5.11)
$$

Thus, equivalently,

$$
\begin{aligned}
b_1 &= \frac{1}{3}\Bigg[11N - n_{cp}\Big\{(N+2) + p(N-2)\Big\}\Bigg] \\
&= \frac{1}{3}\Bigg[11N - 2n_{cp}\Big(\frac{N^2-8}{N-4}\Big)\Bigg] \qquad (5.12)
\end{aligned}
$$

and

$$
\begin{aligned}
b_2 &= \frac{1}{3}\Bigg[34N^2 - n_{cp}\Bigg\{\Big(5N + 3\frac{(N+2)(N-1)}{N}\Big)(N+2) \\
&\quad +p\Big(5N + 3\frac{(N-2)(N+1)}{N}\Big)(N-2)\Bigg\}\Bigg] \\
&= \frac{1}{3}\Bigg[34N^2 - 8n_{cp}\Big\{\frac{2N^4 - 19N^2 + 12}{N(N-4)}\Big\}\Bigg] \ , \qquad (5.13)
\end{aligned}
$$

where it is understood that, since n_S is taken to have its minimal value of 1, the value of N is restricted so that p is a (positive) integer. The requirement of asymptotic freedom, i.e., $b_1 > 0$, implies that n_{cp} is bounded above according to

$$
n_{cp} < \frac{11N(N-4)}{2(N^2-8)} \ . \qquad (5.14)
$$

As a rational number, this upper bound has the respective values (quoted to the indicated floating-point accuracy) 1.62, 2.36, 3.14, and 3.88 for $N = 5$, 6, 8, 12. Therefore, on the integers, we have the upper bounds

$$
n_{cp} \le \begin{cases} 1 & \text{for } N = 5 \\ 2 & \text{for } N = 6 \\ 3 & \text{for } N = 8, \ 12 \end{cases} \ . \qquad (5.15)
$$

Thus, the full set of (anomaly-free) asymptotically free $S\bar{A}$ chiral gauge theories of this type, $(N; n_S, n_{\bar{A}}) = (N; 1, p)$ and $(N; n_{cp}, n_{cp}p)$ with integer p, includes, in addition to the minimal set (5.7), also the additional theories with an n_{cp}-fold replication of the set (5.7), namely

$$(N; n_S, n_{\bar{A}}) = (6; 2, 10),\ (8; 2, 6),\ (8; 3, 9),\ (12; 2, 4),\ (12; 3, 6)\ . \tag{5.16}$$

There are also (asymptotically free) solutions of the anomaly cancellation condition (5.3) with nonminimal values $n_S > 1$ that are not of the form of simple replications of the minimal set (5.7), i.e., for which p is a (positive) rational, but not integer, number. We find that there are seven such solutions, namely

$$\begin{aligned}(N; n_S, n_{\bar{A}}) \quad = \quad & (10; 3, 7),\ (16; 3, 5),\ (20; 2, 3),\ (20; 4, 6), \\ & (28; 3, 4),\ (36; 4, 5),\ (44; 5, 6)\ . \end{aligned} \tag{5.17}$$

Thus, we find that there are sixteen $S\bar{A}$ anomaly-free asymptotically free chiral gauge theories; these consist of the four minimal ones of the form $(N; 1, p)$ in Eq. (5.7), the five theories of the form $(N; n_{cp}, n_{cp}p)$ in Eq. (5.16), and the seven additional ones in Eq. (5.17) with rational, but non-integral p. As noted in the introduction, a striking feature of this family of $S\bar{A}$ chiral gauge theories is that the combined requirements of anomaly cancellation and asymptotic freedom yields only a finite set of solutions, in contrast to generic families of chiral gauge theories that have been studied in the past, such as $S+(N+4)\bar{F}$ and $A+(N-4)\bar{F}$, and extensions of these with vectorlike subsectors, which allow, respectively, the infinite ranges $N \geq 3$ and $N \geq 5$.

We label the fermion fields in a given $S\bar{A}$ theory as

$$S_i:\ \psi^{ab}_{i,L} = \psi^{ba}_{i,L}\ , \quad 1 \leq i \leq n_S \tag{5.18}$$

and

$$\bar{A}_j:\ \chi_{ab,j,L} = -\chi_{ba,j,L}, \quad 1 \leq j \leq n_{\bar{A}}\ , \tag{5.19}$$

where $a,\ b$ are SU(N) gauge indices and $i,\ j$ are flavor indices.

5.1.2 Global Flavor Symmetry

The classical global flavor (fl) symmetry group of the $(N; n_S, n_{\bar{A}})$ $S\bar{A}$ theory is

$$G_{fl,cl} = \mathrm{U}(n_s) \otimes \mathrm{U}(n_{\bar{A}}) = \begin{cases} \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)_S \otimes \mathrm{U}(1)_{\bar{A}} & \text{if } n_s = 1 \\ \mathrm{SU}(n_S) \otimes \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)_S \otimes \mathrm{U}(1)_{\bar{A}} & \text{if } n_s \geq 2 \end{cases} .$$

The operation of the elements of these global groups on the fermion fields is as follows. For fixed SU(N) group indices a, b, the theory is invariant under the action of an element $\mathcal{U}_S \in \mathrm{U}(n_S)$ on the n_S-dimensional vector $(\psi^{ab}_{1,L}, \psi^{ab}_{2,L}, ...\psi^{ab}_{n_S,L})$

$$\psi^{ab}_{i,L} \to \sum_{j=1}^{n_S} (\mathcal{U}_S)_{ij} \psi^{ab}_{j,L} \tag{5.20}$$

and separately under the action of an element of an element $\mathcal{U}_{\bar{A}} \in \mathrm{U}(n_{\bar{A}})$ on the $n_{\bar{A}}$-dimensional vector $(\chi_{ab,1,L}, \chi_{ab,2,L}, ...\chi_{ab,n_{\bar{A}},L})$

$$\chi_{ab,i,L} \to \sum_{j=1}^{n_{\bar{A}}} (\mathcal{U}_{\bar{A}})_{ij} \chi_{ab,j,L} \ . \tag{5.21}$$

The U(1)$_S$ and U(1)$_{\bar{A}}$ global symmetries are both broken by SU(N) instantons [6]. As before in our analysis of different chiral gauge theories [22], we define a vector whose components are comprised of the instanton-generated contributions to the breaking of these symmetries. In the basis $(S, \bar{A})$, this vector is

$$\begin{aligned} \vec{v} &= \Big(n_S T(S),\ n_{\bar{A}} T(\bar{A})\Big) = n_{np}\bigg(\frac{N+2}{2},\ p\frac{N-2}{2}\bigg) \\ &= \frac{n_{np}}{2}\bigg(N+2,\ \frac{(N+4)(N-2)}{N-4}\bigg) \ . \end{aligned} \tag{5.22}$$

We can construct one linear combination of the two original currents that is conserved in the presence of SU(N) instantons. We denote the corresponding global U(1) flavor symmetry as U(1)$'$ and the fermion charges under this U(1)$'$ as

$$\vec{Q}' = \Big(Q'_S,\ Q'_{\bar{A}}\Big) \ . \tag{5.23}$$

The U(1)$'$ current is conserved if and only if

$$\sum_f n_f T(R_f)\, Q'_f = \vec{v} \cdot \vec{Q}' = 0 \ . \tag{5.24}$$

Clearly, this condition determines the vector $\vec{Q}'$ only up to an overall multiplicative constant. A solution is

$$\vec{Q}' = \Big((N-2)(N+4), \ -(N+2)(N-4) \Big) \ . \tag{5.25}$$

The actual global chiral flavor symmetry group (preserved in the presence of instantons) is then

$$G_{fl} = \begin{cases} \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)' & \text{if } n_S = 1 \\ \mathrm{SU}(n_S) \otimes \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)' & \text{if } n_S \geq 2 \end{cases} \ . \tag{5.26}$$

So far, our discussion of global flavor symmetries applies generally to all of the $S\bar{A}$ chiral gauge theories. We next determine the most attractive channel for fermion condensation, which differs for $N = 5$ and $N \geq 6$, and then proceed with analyses of specific $S\bar{A}$ theories.

5.1.3 Fermion Condensation Channels

For $N \neq 5$, the most attractive channel for the formation of a bilinear fermion condensate in a $(N; n_S, n_{\bar{A}})$ $S\bar{A}$ chiral gauge theory is

$$S \times \bar{A} \to adj \ , \tag{5.27}$$

where *adj* denotes the adjoint representation of SU(N). This has

$$\Delta C_2 = \frac{(N+2)(N-2)}{N} \quad \text{for } S \times \bar{A} \to adj \ . \tag{5.28}$$

Substituting this expression for ΔC_2 into Eq. (3.4) for the estimate of the minimum critical coupling for condensation in this channel, we obtain

$$\alpha_{cr} \simeq \frac{2\pi N}{3(N+2)(N-2)} \quad \text{for } S \times \bar{A} \to adj \ . \tag{5.29}$$

In general, there are several stages of fermion condensation, as will be evident in the analyses of specific theories below. In the $S\bar{A}$ theory with $G = \mathrm{SU}(5)$, the most attractive channel is $\bar{A} \times \bar{A} \to F$ instead of (5.27) and will be discussed in the section devoted to this theory.

5.2 $S\bar{A}$ Theory with $G = \mathrm{SU}(5)$

5.2.1 General

Our SU(5) $S\bar{A}$ theory has $(n_S, n_{\bar{A}}) = (1, 9)$. Since $n_S = 1$ for this theory, we use a simplified notation without the flavor index on the S field, namely $\psi^{ab}_{i=1,L} \equiv \psi^{ab}_L$. We recall that the $S = \square\!\square$ and $\bar{A} = \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ representations of SU(5) have dimensionalities 15 and 10, respectively, and we shall equivalently refer to them in this section by these dimensionalities. From Eq. (5.26), this theory has a (nonanomalous) global flavor symmetry

$$G_{fl} = \mathrm{SU}(9)_{\bar{A}} \otimes \mathrm{U}(1)' \ . \tag{5.30}$$

We have not found SU(5)-gauge-singlet composite-fermion operators that satisfy the 't Hooft anomaly matching conditions for this theory. Indeed, the minimal fermionic operator products, such as $S^{ab}\bar{A}_{bc}S^{cd}$, $\epsilon^{abcde}\bar{A}_{ab}\bar{A}_{cd}\bar{A}_{ef}$, are not SU(5) gauge singlets.

We list the values of the first two coefficients of the beta function for this theory in Table 5.1. This beta function has an IR zero which occurs, at the two-loop level, at a value $\alpha_{IR,2\ell} = 0.645$. As will be discussed further below, we find that this value is close to an estimate of the minimal critical value, α_{cr} for the formation of a bilinear fermion condensate in the most attractive channel, with associated spontaneous chiral symmetry breaking. Consequently, we shall analyze both possibilities (i.e., retaining or breaking chiral symmetry) for the UV to IR evolution of this SU(5) $S\bar{A}$ theory.

The most attractive channel for condensation is

$$\text{MAC for } (5;1,9): \quad \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \times \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \to \square, \ \textit{i.e.}, \ \bar{A} \times \bar{A} \to F \ , \tag{5.31}$$

where $F = \square$ is the fundamental representation. Equivalently, in terms of dimensionalities, this is the channel $\overline{10} \times \overline{10} \to 5$. Since $C_2(A_2) = 18/5$ and $C_2(F) = 12/5$ for SU(5), the measure of attractiveness for this channel is

$$\Delta C_2 = \frac{24}{5} \quad \text{for } \overline{10} \times \overline{10} \to 5 \ . \tag{5.32}$$

(The next-most attractive channel is $S \times \bar{A} \to$ adj, with $\Delta C_2 = 21/5$.) The rough Schwinger-Dyson estimate of the critical coupling for condensate formation in the $\overline{10} \times \overline{10} \to 5$ channel is $\alpha_{cr} \sim 5\pi/36 = 0.44$. To compare $\alpha_{IR,2\ell}$ with α_{cr}, we use the ratio ρ defined in Eq. (3.5). We calculate $\rho = 1.5$

Table 5.1: Properties of SU(N) $S\bar{A}$ chiral gauge theories with (i) minimal fermion content $n_S = 1$ and $n_{\bar{A}} = p = (N+4)/(N-4)$ and (ii) n_{cp}-fold replicated fermion content $n_s = n_{cp}$ and $n_{\bar{A}} = n_{cp}p$. The quantities listed are $(N; n_S, n_{\bar{A}})$, p, n_{cp}, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the relevant first condensation channel, and the ratio ρ given by Eq. (3.5). The dash notation $-$ means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, where SχSB indicates spontaneously broken chiral symmetry, χS indicates a chirally symmetric behavior, and ESR stands for "either symmetry realization", χS or SχSB. See text for further discussion of descendant theories.

$(N; n_S, n_{\bar{A}})$	p	n_{cp}	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{IR,2\ell}$	$\alpha_{cr,ch}$	ρ_{IR}	comment
(5;1,9)	9	1	0.5570	-0.8638	0.645	0.44	1.5	ESR
(6;1,5)	5	1	1.008	-0.1182	8.53	0.39	22	SχSB
(8;1,3)	3	1	1.592	0.9056	$-$	0.28	$-$	SχSB
(12;1,2)	2	1	2.600	3.519	$-$	0.18	$-$	SχSB
(6;2,10)	5	2	0.2653	-2.820	0.0941	0.39	0.24	χS
(8;2,6)	3	2	0.8488	-2.782	0.3051	0.28	1.1	ESR
(12;2,4)	2	2	1.698	-3.297	0.5149	0.18	2.9	SχSB
(8;3,9)	3	3	0.1061	-6.470	0.0164	0.28	0.059	χS
(12;3,6)	2	3	0.7958	-10.113	0.07869	0.18	0.44	ESR

for the channel (5.31). This value of ρ is close enough to unity that we cannot make a definite conclusion concerning the presence or absence of spontaneous chiral symmetry breaking. There are thus two possibilities for the first step in the UV to IR evolution of this SU(5) $S\bar{A}$ theory, and we investigate both of these.

5.2.2 Evolution of SU(5) $S\bar{A}$ Theory to a Non-Abelian Coulomb Phase in the IR

First, the SU(5) (5;1,9) $S\bar{A}$ theory might evolve downward in μ without any spontaneous chiral symmetry breaking, yielding a (deconfined) non-Abelian Coulomb phase (NACP) in the infrared. We denote this possibility as

$$(5;1,9): \quad \text{UV} \to \text{IR} \ \ \text{NACP} \ . \tag{5.33}$$

In this case, the global flavor symmetry in the IR is the same as in the UV, namely (5.30).

5.2.3 Dynamical Breaking of SU(5) to SU(4) Gauge Symmetry

Second, the gauge coupling of the (5;1,9) SU(5) $S\bar{A}$ theory might become sufficiently strong to lead to nonperturbative behavior. Since we have not found gauge-singlet operator products that satisfy 't Hooft anomaly matching conditions in this SU(5) theory, we infer that this nonperturbative behavior would lead to the formation of a bilinear fermion condensate, breaking the SU(5) gauge symmetry. We denote this possibility as

$$(5;1,9): \quad \text{UV} \to \text{IR}: \quad S\chi SB \implies \text{SU}(4) \ . \tag{5.34}$$

We proceed to analyze this possibility. Thus, we assume that as the reference Euclidean momentum scale μ decreases below a value that we denote Λ_5, the gauge coupling becomes large enough to form a bilinear condensate in the most attractive channel, $\bar{A} \times \bar{A} \to F$, Eq. (5.31), dynamically breaking the SU(5) gauge symmetry to SU(4) (and also breaking the global flavor symmetry (5.30)). The associated fermion condensate is of the form $\langle \epsilon^{abdef} \chi^T_{bd,i,L} C \chi_{ef,j,L} \rangle$, where C is the charge-conjugation Dirac matrix. With no loss of generality, we may choose the uncontracted index a to be $a = 5$. By a vacuum alignment argument similar to that used in [1], we infer that the actual condensates are of the form

$$\langle \epsilon^{5bdef} \chi^T_{bd,j,L} C \chi_{ef,j,L} \rangle \propto \Big[\langle \chi^T_{12,j,L} C \chi_{34,j,L} \rangle - \langle \chi^T_{13,j,L} C \chi_{24,j,L} \rangle + \langle \chi^T_{14,j,L} C \chi_{23,j,L} \rangle \Big] \tag{5.35}$$

for $1 \le j \le 9$. Since the gauge interaction is independent of the flavor index, it follows that these condensates have a common value independent of the flavor index, j. The fermions involved in these condensates thus gain a common dynamical mass of order Λ_5 and are integrated out of the low-energy effective field theory applicable at energy scales $\mu < \Lambda_5$. When SU(N) breaks to SU($N-1$) there are $2N-1$ gauge bosons in the coset SU(N)/SU($N-1$), corresponding to the broken generators. Here, with $N = 5$, there are nine gauge bosons in the coset SU(5)/SU(4), and these also gain masses of order Λ_5.

5.2.4 Analysis of SU(4) Descendant Theory

Since the low-energy effective field theory resulting as a descendant from the breaking of the SU(5) $S\bar{A}$ gauge symmetry is invariant under an SU(4) gauge symmetry, in order to analyze it, we decompose the remaining massless fermions into SU(4) representations. For this purpose, we make use of the following general results for an SU(N) group:

$$\square\!\square_{\,\mathrm{SU}(N)} = [\, \square\!\square + \square + 1 \,]_{\,\mathrm{SU}(N-1)} \ , \tag{5.36}$$

where 1 denotes a singlet, and, for $N \geq 4$, and

$$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\,\mathrm{SU}(N)} = [\, \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} + \bar{\square} \,]_{\,\mathrm{SU}(N-1)} \ . \tag{5.37}$$

Here, in terms of dimensionalities, these decompositions read

$$15_{\mathrm{SU}(5)} = 10_{\mathrm{SU}(4)} + 4_{\mathrm{SU}(4)} + 1 \tag{5.38}$$

and

$$\overline{10}_{\mathrm{SU}(5)} = 6_{\mathrm{SU}(4)} + \bar{4}_{\mathrm{SU}(4)} \ . \tag{5.39}$$

Note that $6_{\mathrm{SU}(4)}$ is self-conjugate, i.e., $6_{\mathrm{SU}(4)} \approx \bar{6}_{\mathrm{SU}(4)}$. The massless SU(4)-nonsinglet fermions in the SU(4) theory thus consist of ψ_L^{ab} with $1 \leq a,\ b \leq 4$, ψ_L^{a5} with $1 \leq a \leq 4$, and $\chi_{a5,j,L}$ with $1 \leq a \leq 4$ and $1 \leq j \leq 9$. In terms of Young tableaux, these are $\square\!\square + \square + 9\bar{\square}$ under SU(4), or in equivalent notation, the theory is

$$\mathrm{SU}(4), \ \text{fermions}: \ S + F + 9\bar{F} = S + 8\bar{F} + 1\{F + \bar{F}\} \ . \tag{5.40}$$

Thus the SU(4)-nonsinglet fermion content of the theory is precisely the $N = 4$, $p = 1$ special case of the Sp model presented in [11] and further analyzed in [12, 22, 82], so we can apply the results from these previous studies here. This SU(4) theory also contains the massless SU(4)-singlet chiral fermion ψ_L^{55} inherited from the SU(5) theory, but this does not affect the SU(4) dynamics. We recall that the Sp model is defined by the gauge group and chiral fermion content

$$Sp: \ G = \mathrm{SU}(N), \ \text{fermions}: \ S + (N+4)\bar{F} + p\{F + \bar{F}\} \tag{5.41}$$

where the first part, $S + (N+4)\bar{F}$ is irreducibly chiral and the second part is a vectorlike subsector consisting of p copies of $\{F + \bar{F}\}$. As dictated by our theorem proved above, this SU(4) descendant theory is anomaly-free. This is evident from the count

$$\mathcal{A}(\square\!\square_{\,\mathrm{SU}(4)}) \ + \ \mathcal{A}(\square_{\,\mathrm{SU}(4)}) - 9\,\mathcal{A}(\square_{\,\mathrm{SU}(4)})$$

$$= \quad 8+1-9=0 \ . \tag{5.42}$$

The first two coefficients of the beta function of this SU(4) low-energy effective field theory have the same sign (explicitly, $\bar{b}_1 = 0.7427$ and $\bar{b}_2 = 0.1831$), so this beta function has no IR zero at the maximal scheme-independent, two-loop order. Thus, as the Euclidean momentum scale μ decreases below Λ_5, the SU(4) gauge coupling inherited from the original SU(5) theory continues to increase until it exceeds the region where it can be described by the perturbative beta function. There are then several possibilities for the next stage of RG evolution to lower scales. We discuss these next.

Confinement in SU(4) Theory with Massless Composite Fermions

The first of these possibilities for the SU(4) theory is present because of the fact that (for general values of N and p where there is confinement) the Sp model satisfies the 't Hooft anomaly-matching conditions [11, 22, 82]. Owing to this, as the gauge coupling continues to increase in the infrared, the gauge interaction could confine the (massless) SU(4)-nonsinglet fermions, producing massless spin 1/2 composite fermions as well as massive SU(4)-singlet hadrons (mesons, glueballs, and mass eigenstates that are linear combinations of mesons and glueballs). The massless fermion spectrum would also contain the SU(4)-singlet chiral fermion ψ_L^{55} from the original SU(5) $S\bar{A}$ theory.

Formation of Fermion Condensates Breaking SU(4) Gauge Symmetry

The second of these possibilities for the UV to IR evolution of the SU(4) low-energy effective field theory resulting from the breaking of the SU(5) $S\bar{A}$ theory is further fermion condensation in the most attractive channel in this SU(4) theory. The MAC is the channel $\square\!\square \times \bar{\square} \to \square$, i.e.,

$$S \times \bar{F} \to F \ . \tag{5.43}$$

The next-most attractive channel is $F \times \bar{F} \to 1$, with

$$\Delta C_2 = 2C_2(F) = \frac{N^2-1}{N} \ . \tag{5.44}$$

The fact that the $S \times \bar{F} \to F$ channel is the MAC is evident from the property that it has a larger ΔC_2 value than the $F \times \bar{F} \to 1$ channel:

$$\frac{(N+2)(N-1)}{N} - \frac{N^2-1}{N} = \frac{N-1}{N} > 0 \ . \tag{5.45}$$

For generality, we discuss the physics of the $S \times \bar{F} \to F$ channel for general N, although our specific application will be to $N = 4$. The attractiveness measure for this channel is

$$\Delta C_2 = C_2(S) = \frac{(N+2)(N-1)}{N} \quad \text{for } S \times \bar{F} \to F \ . \tag{5.46}$$

Substituting this into Eq. (3.4) for the estimate of the minimum critical coupling for condensation in this channel, we obtain

$$\alpha_{cr} \simeq \frac{2\pi N}{3(N+2)(N-1)} \quad \text{for } S \times \bar{F} \to F \ . \tag{5.47}$$

For present case of $N = 4$, this yields the estimate $\Delta C_2 = 4.5$. We denote the Euclidean scale μ at which the running coupling $\alpha(\mu)$ exceeds the critical value for condensation in this MAC as Λ_4. The condensation (5.43) breaks SU(4) to SU(3). The associated condensate has the general form $\langle \sum_{b=1}^4 \psi_L^{ab\ T} C \chi_{5b,j,L} \rangle$. Without loss of generality, we can denote the breaking axis as $a = 4$ and label the copy (flavor) index of the $\bar{F}$ fermion $\chi_{5b,j,L}$ involved in this condensate as $j = 9$, so that the condensate is

$$\langle \sum_{b=1}^{4} \psi_L^{4b\ T} C \chi_{b5,9,L} \rangle \ . \tag{5.48}$$

The fermions ψ_L^{b4} and $\chi_{b5,9,L}$ with $1 \le b \le 4$ involved in this condensate thus get common dynamical masses of order Λ_4. The seven gauge bosons in the coset SU(4)/SU(3) also get masses of order Λ_4. These fermions and bosons are integrated out of the low-energy effective field theory that is operative for scales $\mu < \Lambda_4$.

This low-energy effective field theory is invariant under an (anomaly-free) SU(3) gauge symmetry and contains the massless SU(3)-nonsinglet chiral fermions ψ_L^{ab} with $1 \le a,\ b \le 3$, transforming as $S = \square\square$ of SU(3), ψ_L^{a5}, with $1 \le a \le 3$, transforming as $F = \square$ of SU(3), and the $\chi_{a5,j,L}$ with $1 \le a \le 3$ and $1 \le j \le 8$; that is,

$$G = \mathrm{SU}(3), \ \text{fermions}: \quad S + F + 8\,\bar{F} = S + 7\bar{F} + 1\{F + \bar{F}\} \tag{5.49}$$

The SU(3)-nonsinglet fermion content of this theory is the $N = 3$, $p = 1$ special case of the Sp model (5.41). This SU(3) theory also contains a number of massless SU(3)-singlet chiral fermions. In addition to the ψ_L^{55} SU(4)-singlet fermion remaining from the SU(5) $\to$ SU(4) breaking at the higher scale Λ_5, there are also nine massless SU(3)-singlet fermions remaining from the SU(4) $\to$ SU(3) breaking at Λ_4, namely ψ_L^{45} and the $\chi_{45,j,L}$ with $1 \le j \le 8$.

As discussed in [11, 22, 82], the further evolution into the infrared of this SU(3) Sp model might lead to confinement with resultant massless composite fermions or to further condensation in the most attractive channel, which is $S \times \bar{F} \to F$, breaking SU(3) to SU(2) and then breaking SU(2) completely. In the latter case, the full sequence of gauge symmetry breaking of (5;1,9) theory would be as follows: $\bar{A} \times \bar{A} \to F$, breaking SU(5) to SU(4), followed in the resultant SU(4) descendant theory by the condensation $S \times \bar{F} \to F$, breaking SU(3) to SU(2), followed again by condensation in the respective $S \times \bar{F} \to F$ channel, breaking SU(2) completely.

5.3 $S\bar{A}$ models with $N \ge 6$

5.3.1 General Analysis

We next proceed to analyze the $S\bar{A}$ models $(N; n_S, n_{\bar{A}})$ with $N \ge 6$. In contrast to the SU(5) $S\bar{A}$ theory $(5; 1, 9)$, if $N \ge 6$, the most attractive channel for bilinear fermion condensation is $S \times \bar{A} \to adj$, as given in Eq. (5.27): This condensation produces, as the first stage of dynamical gauge symmetry breaking, the pattern

$$\mathrm{SU}(N) \to \mathrm{SU}(N-1) \otimes \mathrm{U}(1) \ . \tag{5.50}$$

The values of ΔC_2 and α_{cr} for this channel were given in Eqs. (5.28) and (5.29). The resultant estimates for α_{cr} for condensation in this channel in specific $(N; n_S, n_{\bar{A}})$ models with $N \ge 6$ are listed in Tables 5.1 and 5.2. In these tables we also list the (reduced) beta function coefficients $\bar{b}_1$ and $\bar{b}_2$, the resultant IR zero in the two-loop beta function, if it exists, and the ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr}$ from Eq. (3.5). In cases where the beta function has no IR zero, the coupling increases with decreasing reference scale μ until it exceeds the perturbatively calculable regime. For a generic $S\bar{A}$ $(N; n_S, n_{\bar{A}})$ theory, we do not find solutions for 't Hooft anomaly matching, although, as will be discussed later, in certain cases, resultant low-energy effective descendant

Table 5.2: Properties of SU(N) $S\bar{A}$ chiral gauge theories with other $(N; n_S, n_{\bar{A}})$ than those in Table 5.1. The quantities listed are $(N; n_S, n_{\bar{A}})$, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the relevant first condensation channel, and the ratio ρ given by Eq. (3.5). The dash notation $-$ means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, where SχSB indicates spontaneously broken chiral symmetry, χS indicates a chirally symmetric behavior, and ESR stands for "either symmetry realization", χS or SχSB. See text for further discussion of descendant theories.

$(N; n_S, n_{\bar{A}})$	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{IR,2\ell}$	$\alpha_{cr,ch}$	ρ_{IR}	comment
(10;3,7)	0.4775	-8.116	0.0588	0.22	0.27	χS
(16;3,5)	1.379	-14.931	0.0924	0.13	0.69	ESR
(20;2,3)	3.236	-4.265	0.759	0.11	7.2	SχSB
(20;4,6)	0.6366	-37.238	0.0171	0.11	0.16	χS
(28;3,4)	3.024	-35.286	0.0857	0.075	1.1	SχSB
(36;4,5)	1.963	-102.512	0.0191	0.058	0.33	χS
(44;5,6)	0.05305	-218.913	0.242e-3	0.048	0.0051	χS

field theories with different fermion content (e.g., the SU(4) Sp model below), do satisfy these matching conditions. Thus, as regards the initial UV to IR evolution of the $(N; n_S, n_{\bar{A}})$ theory, if the gauge coupling becomes sufficiently strong, one expects fermion condensation. The resulting expectations for whether or not fermion condensation and associated spontaneous chiral symmetry breaking occur are listed for the sixteen $S\bar{A}$ theories in Tables 5.1 and 5.2.

Flow to Chirally Symmetric Non-Abelian Coulomb Phase in IR

Referring to these Tables 5.1 and 5.2, in the six cases where the value of ρ is substantially less than unity, we infer that the theory is likely to evolve smoothly from the UV to a (deconfined) chirally symmetric non-Abelian Coulomb phase in the IR. Explicitly, we infer that this IR behavior occurs for the (6;2,10), (8;3,9), (10;3,7), (20;4,6), (36;4,5), and (44;5,6) $S\bar{A}$ theories.

Flow to IR with Spontaneous Chiral Symmetry Breaking

We next discuss the situation in which, as the reference scale μ decreases from the UV to the IR, the coupling becomes large enough so that nonperturbative behavior occurs. As noted, in the absence of sets of fermionic operator products that yield solutions to 't Hooft anomaly matching conditions, one infers that this nonperturbative behavior entails fermion condensation and associated spontaneous breaking of the SU(N) gauge symmetry (although after some stage(s) of such symmetry breaking, a low-energy descendant theory may satisfy these matching conditions). For technical simplicity, we restrict our discussion to the minimal theories $(N;1,p)$; corresponding analyses can be given for the other $(N;n_S,n_{\bar{A}})$ models. As is evident from Table 5.1, all three of of the $(N;1,p)$ theories with $N \geq 6$, namely (6;1,5), (8;1,3), and (12;1,2) have the property that the gauge coupling becomes sufficiently strong to produce further bilinear fermion condensation. As before, we denote the scale where this occurs as Λ_N. A vacuum alignment argument implies that the symmetry breaking is such as to leave the largest residual symmetry. This implies that the condensate breaks the original SU(N) gauge symmetry to SU($N-1$) $\otimes$ U(1). Without loss of generality, we take the breaking direction in SU(N) to be $a = N$. To show how this occurs, we recall the decompositions of $\square\square_{\,\mathrm{SU}(N)}$ and $\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\,\mathrm{SU}(N)}$ under SU($N-1$) given, respectively, in Eqs. (5.36) and (5.37) above. Using these decompositions, we have

$$\square\square_{\,\mathrm{SU}(N)} \times \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\,\mathrm{SU}(N)} = \left(\square\square_{\,\mathrm{SU}(N-1)} + \square_{\,\mathrm{SU}(N-1)} + 1\right) \times \left(\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\,\mathrm{SU}(N-1)} + \overline{\square}_{\,\mathrm{SU}(N-1)}\right) . \tag{5.51}$$

Among the various products, we see that $\square_{\,\mathrm{SU}(N-1)} \times \overline{\square}_{\,\mathrm{SU}(N-1)}$ yields a singlet of SU($N-1$) and hence is favored by the vacuum alignment argument. The associated condensate thus has the form (with no sum on a)

$$\langle T^a_a \rangle = \begin{cases} \kappa & \text{for } 1 \leq a \leq N-1 \\ -\kappa(N-1) & \text{for } a = N \end{cases} \tag{5.52}$$

where κ is a constant. Thus, in terms of the fermion fields, the $\langle T^a_a \rangle$ condensate is of the form $\langle T^a_a \rangle = \langle \psi^{ad\;T}_L C \chi_{da,n_{\bar{A}},L} \rangle$ (with no sum on a or d). The fermions involved in this condensate gain dynamical masses of order Λ_N. The $2N$ gauge bosons in the coset SU(N)/[SU($N-1$) $\otimes$ U(1)] also gain masses of order Λ_N. In the low-energy effective field theory that is applicable at scales $\mu < \Lambda_N$, one thus integrates out these fields with masses $\sim \Lambda_N$.

The massless nonsinglet chiral fermion content of the resultant low-energy effective field theory with $\mathrm{SU}(N-1) \otimes \mathrm{U}(1)$ gauge invariance thus consists of the $\square\square_{\,\mathrm{SU}(N-1)}$, p copies of the $\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\,\mathrm{SU}(N-1)}$, and $p-1$ copies of the $\bar{\square}_{\,\mathrm{SU}(N-1)}$, with corresponding U(1) charges. We summarize this $\mathrm{SU}(N-1)$-nonsinglet content as

$$\mathrm{SU}(N-1): \quad \text{fermions}: \; S + p\,\bar{A} + (p-1)\bar{F} \tag{5.53}$$

where S, $\bar{A}$, and $\bar{F}$ refer to the $\mathrm{SU}(N-1)$ gauge symmetry. This $\mathrm{SU}(N-1)$ effective theory also contains the massless $\mathrm{SU}(N-1)$-singlet fermion ψ_L^{NN}. As guaranteed by the theorem proved above, this descendant theory is free of anomalies in gauged currents. This is evident for the $\mathrm{SU}(N-1)^3$ triangle anomaly, for example, since this is given (with $M \equiv N-1$) by:

$$\begin{aligned} \mathrm{SU}(N-1)^3 \; \mathcal{A} &= \mathcal{A}(S) + p\mathcal{A}(\bar{A}) + (p-1)\mathcal{A}(\bar{F}) \\ &= (M+4) - p(M-4) - (p-1) = 0 \;, \end{aligned} \tag{5.54}$$

where the last line follows upon substitution of p from Eq. (5.5). Similar cancellations hold for the $\mathrm{SU}(N-1)^2\,\mathrm{U}(1)$ and $\mathrm{U}(1)^3$ anomalies.

5.3.2 SU(6) Theory with $n_S = 1$, $n_{\bar{A}} = 5$

The renormalization-group evolution of a $(N; n_S, n_{\bar{A}})$ theory into the infrared depends on the specific theory. For definiteness, we shall focus on the (6;1,5) theory for our further discussion. We list the values of the first two coefficients of the beta function for this theory in Table 5.1. As before, since $n_S = 1$ for this theory, we use a simplified notation without the flavor index on the S field, namely $\psi^{ab}_{i=1,L} \equiv \psi^{ab}_L$. Our discussion for general $N \ge 6$ applies, in particular, to this theory.

Initial Condensation and Breaking of SU(6) to $\mathrm{SU}(5) \otimes \mathrm{U}(1)$

Since the ratio ρ is substantially larger than unity (see Table 5.1) and since we have not found composite fermion operators that satisfy the 't Hooft anomaly matching conditions, we infer that as the reference scale μ decreases from the UV to the IR, the gauge interaction produces a bilinear fermion condensate in the $S \times \bar{A} \to adj$ channel. Using the notation introduced above, this occurs at a scale denoted Λ_6. By convention, we take the breaking direction as $a = 6$ and the copy (flavor) label of the $\bar{A}_2$ fermion involved to be $j = p = 5$. In the notation of Eq. (5.52), the condensate can then be written as

$$\langle T^a_a \rangle = \langle \psi_L^{a6\ T} \chi_{6a,5,L} \rangle \tag{5.55}$$

where $1 \le a \le 5$, and there is no sum on a. The fermions involved in this condensate gain dynamical masses of order Λ_6.

Analysis of Descendant $\mathrm{SU}(5) \otimes \mathrm{U}(1)$ **Theory**

We next consider the descendant $\mathrm{SU}(5) \otimes \mathrm{U}(1)$ theory that emerges from the self-breaking of the SU(6) theory at Λ_6. The relevant decomposition of the SU(6) S and $\bar{A}$ representations under $\mathrm{SU}(5) \otimes \mathrm{U}(1)$, are indicated as follows, in terms of Young tableaux and SU(5) dimensionalities, with U(1) charges given as subscripts (normalized according to the conventions of [104]):

$$\begin{aligned} \square\!\square_{\ \mathrm{SU}(6)} &= [\ \square\!\square + \square + 1\ ]_{\ \mathrm{SU}(5)} \\ &= 15_2 + 5_{-4} + 1_{-10} \end{aligned} \tag{5.56}$$

and

$$\begin{aligned} \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{\ \mathrm{SU}(6)} &= [\ \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} + \bar{\square}\ ]_{\ \mathrm{SU}(5)} \\ &= \overline{10}_{-2} + \bar{5}_4\ . \end{aligned} \tag{5.57}$$

We will again use the shorthand notation $S \equiv S_2$, $\bar{A} \equiv \bar{A}_2$, and $\bar{F}$ for the $\square\!\square$, $\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$ and $\bar{\square}$, where these now refer to SU(5). We will indicate the U(1) charge of the S field as η_S and so forth for the other fermion fields. The massless SU(5)-nonsinglet fermion content in this effective theory is thus

$$\mathrm{SU}(5): \quad \text{fermions}:\ S\ +\ 5\,\bar{A}\ +\ 4\ \bar{F}\ . \tag{5.58}$$

Explicitly, these fermions (with dimensions of the SU(5) representations indicated in parentheses) are

$$S(15): \quad \psi_L^{ab} \text{ with } 1 \le a,\ b \le 5,$$

$$5\ \bar{A}(10): \quad \chi_{ab,j,L} \text{ with } 1 \le a,\ b \le 5 \text{ and } 1 \le j \le 5,$$

$$4\ \bar{F}(5): \quad \chi_{6b,j,L} \text{ with } 1 \le b \le 5 \text{ and } 1 \le j \le 4\ . \tag{5.59}$$

This theory also contains the massless SU(5)-singlet fermion ψ_L^{66} from the original SU(6) theory. From Eq. (5.56), it follows that this fermion has U(1) charge $\eta_1 = -10$.

As an illustration of our theorem, it is instructive to see explicitly the cancellation of contributions to the anomalies in various gauge currents in this SU(5) ⊗ U(1) descendant gauge theory. There are three triangle anomalies that are relevant, namely the $\mathrm{SU}(5)^3$, $\mathrm{SU}(5)^2\,\mathrm{U}(1)$, and $\mathrm{U}(1)^3$ anomalies. We have

$$\begin{aligned} \mathrm{SU}(5)^3\ \mathcal{A} &= \mathcal{A}(S) + 5\,\mathcal{A}(\bar{A}) + 4\,\mathcal{A}(\bar{F}) \\ &= 9 + 5(-1) + 4(-1) = 0 \end{aligned} \tag{5.60}$$

and

$$\begin{aligned} \mathrm{SU}(5)^2\,\mathrm{U}(1)\ \mathcal{A} &= T(S)\eta_S + 5\,T(\bar{A})\eta_{\bar{A}} + 4\,T(\bar{F})\eta_{\bar{F}} \\ &= \frac{7}{2}\,2 + 5\,\frac{3}{2}(-2) + 4\,\frac{1}{2}(4) = 0\ . \end{aligned} \tag{5.61}$$

For the $\mathrm{U}(1)^3$ anomaly cancellation, we must also include the contribution of the SU(5)-singlet fermion ψ_L^{66} since it carries a nonzero U(1) charge:

$$\begin{aligned} & \mathrm{U}(1)^3\ \mathcal{A} = \dim(S)\eta_S^3 + 5\dim(\bar{A})\eta_{\bar{A}}^3 \\ + \quad & 4\dim(\bar{F})\eta_{\bar{F}}^3 + \eta_1^3 \\ = \quad & 15(2^3) + 5(10)(-2)^3 + 4(5)(4^3) + (-10)^3 = 0\ . \end{aligned} \tag{5.62}$$

We also observe that the mixed gauge-gravitational anomaly vanishes:

$$\begin{aligned} & (\mathrm{grav})^2\,U(1)\ \mathcal{A} = \dim(S)\eta_S + 5\dim(\bar{A})\eta_{\bar{A}} \\ + \quad & 4\dim(\bar{F})\eta_{\bar{F}} + \eta_1 \\ = \quad & 15(2) + 5(10)(-2) + 4(5)(4) + (-10) = 0\ . \end{aligned} \tag{5.63}$$

The first two (reduced) coefficients of the SU(5) beta function are $\bar{b}_1 = 0.76925$ and $\bar{b}_2 = -0.22882$, so that this two-loop SU(5) beta function has an IR zero at $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2 = 3.36$. The U(1) beta function is not asymptotically free, so that as the reference scale μ decreases, the running U(1) gauge coupling inherited from the original SU(6) theory decreases. As regards the SU(5) dynamics, the most attractive channel for fermion condensation is $S \times \bar{F} \to F$, with $\Delta C_2 = 28/5$. (The next-most attractive channel is $\bar{A} \times \bar{A} \to F$, with $\Delta C_2 = 24/5$.) Fermion condensation in this most attractive channel causes the gauge symmetry breaking

$$\mathrm{SU}(5) \otimes \mathrm{U}(1) \to \mathrm{SU}(4) \ . \tag{5.64}$$

The fact that the fermion condensate breaks the U(1) gauge symmetry is evident, since $\eta_S + \eta_{\bar{F}} = 6 \neq 0$. The estimated minimum critical coupling for condensation in this MAC to occur is $\alpha_{cr} \simeq 0.38$. Since the ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 9.0$, and since we have not found SU(5) $\otimes$ U(1)-invariant fermionic operator products that satisfy 't Hooft anomaly matching, we anticipate that fermion condensation occurs in this most attractive channel. We denote the scale at which this occurs as Λ_5. The $S\bar{F}$ condensate is of the form $\langle \sum_{b=1}^{5} \psi_L^{ab\ T} C \chi_{b6,j,L} \rangle$. By convention, we denote the breaking direction as $a = 5$ and choose the copy index on the $\chi_{b6,j,L}$ field to be $j = 4$, so that the condensate is

$$\langle \sum_{b=1}^{5} \psi_L^{5b\ T} C \chi_{b6,4,L} \rangle \ . \tag{5.65}$$

The fermions ψ_L^{5b} and $\chi_{b6,4,L}$ with $1 \le b \le 5$ that are involved in this condensate gain dynamical masses of order Λ_5. The ten gauge bosons in the coset [SU(5) $\otimes$ U(1)]/SU(4) also gain masses of order Λ_5. These fields are integrated out in the construction of the SU(4)-invariant low-energy effective field theory applicable at scales $\mu < \Lambda_5$.

Analysis of Descendant SU(4) Theory

The massless SU(4)-nonsinglet chiral fermion content of this effective low-energy theory consists of $\square\square$, 5 copies of $\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$, and eight copies of $\bar{\square}$, i.e.

$$\mathrm{SU}(4): \quad \text{fermions}: \ S + 5\,\bar{A} + 8\,\bar{F} \ . \tag{5.66}$$

Explicitly, these fermions (with dimensions of the SU(5) representations indicated in parentheses) are

$$S(10): \quad \psi_L^{ab} \text{ with } 1 \le a,\ b \le 4$$

$$
\begin{aligned}
5\ \bar{A}(6):&\quad \chi_{ab,j,L} \text{ with } 1 \le a,\ b \le 4 \text{ and } 1 \le j \le 5 \\
5\ \bar{F}(4):&\quad \chi_{5b,j,L} \text{ with } 1 \le b \le 4 \text{ and } 1 \le j \le 5 \\
3\ \bar{F}(4):&\quad \chi_{6b,j,L} \text{ with } 1 \le b \le 4 \text{ and } 1 \le j \le 3\ . \qquad (5.67)
\end{aligned}
$$

This theory also contains the massless SU(4)-singlet fermions ψ_L^{66} and $\chi_{65,j,L}$ with $1 \le j \le 3$.

In accordance with our theorem, we show explicitly that this SU(4) descendant theory is anomaly-free:

$$
\mathrm{SU}(4)^3\ \mathcal{A} = \mathcal{A}(S) + 5\,\mathcal{A}(\bar{A}) + 8\,\mathcal{A}(\bar{F}) = 8 + 0 + 8(-1) = 0\ . \qquad (5.68)
$$

where we have used the fact that the $\overline{\begin{array}{c}\square\\[-4pt]\square\end{array}} = \bar{A}$ representation of SU(4) is self-conjugate.

The first two (reduced) coefficients of the beta function of this SU(4) descendant theory are $\bar{b}_1 = 0.5305$ and $\bar{b}_2 = 0.1224$, with the same sign, so at the maximal scheme-independent, two-loop level, the beta function has no IR zero. Hence, as the reference scale decreases below Λ_5, the SU(4) gauge coupling inherited from the SU(5) theory continues to increase, eventually exceeding the range where it is perturbatively calculable. The most attractive channel for the formation of a bilinear fermion condensate is

$$
\bar{A} \times \bar{A} \to 1\ , \qquad (5.69)
$$

with $\Delta C_2 = 2C_2(A) = 5$. Clearly, this fermion condensation preserves the SU(4) gauge symmetry. The estimated minimal critical coupling for condensation in this channel is $\alpha_{cr} = 2\pi/15 = 0.42$. The associated condensates are of the form $\langle \epsilon^{abde} \chi_{ab,j,L}^T C \chi_{de,k,L} \rangle$, where the copy indices take on values in the interval $1 \le j, k \le 5$. Applying a vacuum alignment argument, we may take $j = k$, so that these condensates are

$$
\langle \epsilon^{abde} \chi_{ab,j,L}^T C \chi_{de,j,L} \rangle \quad \text{for } 1 \le j \le 5 \qquad (5.70)
$$

(where there is no sum on j). These condensates are equal and hence preserve an O(5) isospin symmetry. We denote the scale at which this condensation takes place as $\Lambda_{\bar{A}\bar{A}}$. Owing to this condensation, all of the $\chi_{ab,j,L}$ fields gain masses of order $\Lambda_{\bar{A}\bar{A}}$.

This leaves a descendant (anomaly-free) chiral gauge theory with massless SU(4)-nonsinglet fermion content $S + 8\bar{F}$, given by Eq. (5.67) with the $\bar{A}$ fields removed. This theory has been studied before [11,22,23,82], and we can combine the known results with the new ingredients here for our analysis. The first two (reduced) coefficients in the beta function are $\bar{b}_1 = 0.7958$ and $\bar{b}_2 = 0.2913$, with the same sign, so that this beta function has no IR zero. Hence, as the scale μ decreases below $\Lambda_{\bar{A}\bar{A}}$, the SU(4) gauge coupling continues to increase from its value at $\Lambda_{\bar{A}\bar{A}}$.

Since it is known that this $S + 8\bar{F}$ theory satisfies the 't Hooft anomaly matching conditions [11, 22, 82], one possibility is that it confines without any spontaneous chiral symmetry breaking, producing massless composite fermions and massive hadrons. An alternate type of IR behavior is fermion condensation in the most attractive channel, which is $S \times \bar{F} \to F$, breaking SU(4) to SU(3), followed by further fermion in the respective MAC $S \times \bar{F} \to F$ channels in the descendant SU(3) and SU(2) theories, finally breaking the gauge symmetry completely.

Discussion

It is of interest to contrast our present SU(N) $S\bar{A}$ theories with $N \geq 6$, and hence a most attractive channel of the form $S \times \bar{A} \to adj$, with the theories analyzed in Ref. [27]. One of the purposes of Ref. [27] was to investigate how a fermion condensate transforming as the adjoint representation of a simple SU(N) gauge theory would dynamically break the gauge symmetry, and to contrast this with the types of gauge symmetry breaking patterns that one obtains if one uses a fundamental Higgs field transforming according to the adjoint representation of the SU(N) group. The type of theory considered in [27] had a direct-product gauge group of the form

$$G_{UV} = G \otimes G_b \ , \tag{5.71}$$

where G is a chiral gauge symmetry and G_b is a vectorial gauge symmetry. As constructed, the G_b gauge interaction becomes strong in the infrared and leads to a condensate involving a fermion field transforming as a nonsinglet under both G and G_b, and specifically as the adjoint representation of G, thereby breaking G to a subgroup, H. At the stage where this breaking occurs, the G gauge interaction is still weak. With $G = \text{SU}(N)$ regarded as a hypothetical grand unification group, Ref. [27] addressed the question of what the pattern of induced dynamical breaking of a grand unified theory would be and how it would differ from the pattern obtained with a nonzero vacuum expectation value of a fundamental Higgs field in the adjoint representation. This exploration of possible dynamical symmetry breaking of a grand unified theory is reminiscent of, although different from, the old idea of dynamical breaking of electroweak gauge symmetry by means of a vectorial, strongly coupled, confining gauge theory which would produce bilinear fermion condensates involving fermion(s) that transform under both the electroweak gauge group and the strongly coupled gauge group [15, 105]. In the latter case, the breaking of the electroweak gauge symmetry G_{EW} is caused by a bilinear fermion condensate transforming as the fundamental, rather than adjoint, representation of weak $\mathrm{SU}(2)_L$ with weak hypercharge $Y = 1$.

The difference with respect to our present work is that here we study a chiral gauge theory with a single gauge group rather than a direct product, and the chiral gauge interaction may produce condensates that self-break the strongly coupled chiral gauge symmetry instead of having a weakly coupled chiral gauge symmetry broken by a condensate of fermions that are nonsinglets under both G and G_b. The common feature shared by the dynamical gauge symmetry breaking in studied in [27] and the $S\bar{A}$ theories with $N \geq 6$ here is that the bilinear fermion condensate transforms as an adjoint of the $\mathrm{SU}(N)$ gauge symmetry and breaks it at the highest stage according to the pattern (5.50). To see how this differs with the situation with a Higgs field Φ in the adjoint representation, we recall the Higgs potential (with a $\Phi \to -\Phi$ symmetry imposed for technical simplicity),

$$V = \frac{\mu^2}{2}\mathrm{Tr}(\Phi^2) + \frac{\lambda_1}{4}[\mathrm{Tr}(\Phi^2)]^2 + \frac{\lambda_2}{4}\mathrm{Tr}(\Phi^4) \ , \tag{5.72}$$

where μ^2, λ_1, and λ_2 are real for hermiticity. One chooses $\mu^2 < 0$ to produce the symmetry breaking. Assuming $N \geq 4$, for the comparison here, it follows that the two quartic terms in (5.72) are independent, and the requirement that V be bounded below implies that $\lambda_1 > 0$. This boundedness condition allows λ_2 to take on a restricted range of negative values depending on λ_1 and N, namely [27]

$$-\left(\frac{N(N-1)}{N^2 - 3N + 3}\right)\lambda_1 < \lambda_2 < 0 \ . \tag{5.73}$$

With λ_2 in this interval, V is minimized with a Higgs VEV such that the $\mathrm{SU}(N)$ gauge symmetry is broken according to (5.50). However, if $\lambda_2 >$

0, then V is minimized for a Higgs VEV that yields a different symmetry breaking: if N is even, then the breaking pattern is

$$\mathrm{SU}(N) \to \mathrm{SU}(N/2) \otimes \mathrm{SU}(N/2) \otimes \mathrm{U}(1) \ , \tag{5.74}$$

while if N is odd, then the breaking is

$$\mathrm{SU}(N) \to \mathrm{SU}((N+1)/2) \otimes \mathrm{SU}((N-1)/2) \otimes \mathrm{U}(1) \ . \tag{5.75}$$

This comparison elucidates the difference between the breaking of a gauge symmetry by the VEV of a fundamental Higgs field and the dynamical symmetry breaking by a fermion condensate produced by a strongly coupled gauge interaction.

5.4 Investigation of $S_k\bar{A}_k$ Chiral Gauge Theories with $k \ge 3$

It is natural to ask whether the type of asymptotically free (anomaly-free) chiral gauge theories that we have constructed and studied here with chiral fermions transforming according to the rank-2 symmetric and conjugate antisymmetric representations of SU(N) can be extended to corresponding chiral gauge theories with chiral fermions in the rank-k symmetric and rank-ℓ conjugate antisymmetric representations of SU(N) with $k, \ \ell \ge 3$. We show here that this cannot be done for the diagonal case $k = \ell$ because such theories are not asymptotically free. Thus, our $S\bar{A}$ theories are the unique realization of asymptotically free $S_k \ \bar{A}_\ell$ chiral gauge theories with diagonal $k = \ell \ge 2$.

Let us then consider a chiral gauge theory with chiral fermions transforming according to the rank-k symmetric and conjugate antisymmetric representations of SU(N), denoted as the S_k and $\bar{A}_k$. We denote the number of these fermions as n_{S_k} and $n_{\bar{A}_k}$, respectively, and the theory itself as

$$(N; k; n_{S_k}, n_{\bar{A}_k}) \ . \tag{5.76}$$

The correspondence of this notation with the shorthand notation in the previous part of the text, which studied the $k = 2$ case, is

$$(N; 2; n_{S_2}, n_{\bar{A}_2}) \equiv (N; n_S, n_{\bar{A}}) \ . \tag{5.77}$$

The condition that the theory must be free of any triangle anomaly in gauged currents is

$$n_{S_k}\mathcal{A}(S_k) + n_{\bar{A}_k}\mathcal{A}(\bar{A}_k) = n_{S_k}\mathcal{A}(S_k) - n_{\bar{A}_k}\mathcal{A}(A_k) = 0 \ , \tag{5.78}$$

where $\mathcal{A}(S_k)$ and $\mathcal{A}(A_k)$ are given in Eqs. (A.21) and (A.22) of Appendix A. A solution of this equation has the ratio of copies of fermions in the S_k and $\bar{A}_k$ representations given by

$$\frac{n_{\bar{A}_k}}{n_{S_k}} = \frac{\mathcal{A}(S_k)}{\mathcal{A}(A_k)} \equiv p_k = \frac{(N+2k)(N+k)!(N-k-1)!}{(N-2k)(N+2)!(N-3)!} \ . \tag{5.79}$$

In the case $k=2$ discussed in detail above, $p_k = p$ given in Eq. (5.5). For $k \geq 3$, p_k can also be expressed as

$$p_k = \frac{(N+2k)\Big[\prod_{j=3}^{k}(N+j)\Big]}{(N-2k)\Big[\prod_{j=3}^{k}(N-j)\Big]} \quad \text{for } k \geq 3 \ . \tag{5.80}$$

For example,

$$p_3 = \frac{(N+6)(N+3)}{(N-6)(N-3)} \tag{5.81}$$

and

$$p_4 = \frac{(N+8)(N+3)(N+4)}{(N-8)(N-3)(N-4)} \ . \tag{5.82}$$

For a physical solution, this ratio (5.79) must be positive, which requires that

$$N \geq 2k+1 \ , \tag{5.83}$$

and we restrict N to this range. From Eq. (5.79) it follows that if $k \geq 2$, then $n_{\bar{A}_k} > n_{S_k}$. Therefore the theories of this type with minimal chiral fermion content have the form

$$(N;k;n_S,n_{\bar{A}}) = (N;k;1,p_k) \ , \tag{5.84}$$

with the understanding that p_k must be a (positive) integer. If $k=3$, there are only two solutions of Eq. (5.79) with $n_S = 1$ that satisfy this condition that p_k be an integer, namely

$$(N;3;1,p_3) = \{(9;3;1,10) \ , (12;3;1,5)\} \ . \tag{5.85}$$

The number of solutions decreases as k increases. Thus, if $k = 4$, then there is only one such theory, viz.,

$$(N;4;1,p_4) = \{(10;4;1,39)\} \ , \tag{5.86}$$

and similarly, if $k = 5$, there is only one solution,

$$(N;5;1,p_5) = \{(11;5;1,210)\} \ , \tag{5.87}$$

while we have not found solutions with integer p_k for $k \geq 6$. Theories with n_{cp} copies of this minimal fermion content also satisfy the anomaly cancellation condition (5.78), e.g., if $k = 3$, then the theories $(9;3;n_{cp},10n_{cp})$ and $(12;3;n_{cp},5n_{cp})$ for $n_{cp} \geq 2$ also satisfy the anomaly cancellation condition.

To test whether any of these solutions yield theories that are asymptotically free, we begin by calculating the first coefficient of the beta function for cases with minimal fermion content, with $n_{cp} = 1$, which is

$$b_1 = \frac{1}{3}\Big[11N - 2(T_{S_k} + p_k T_{\bar{A}_k})\Big] \ . \tag{5.88}$$

If $k = 3$, this is

$$\begin{aligned} k=3 \ \rightarrow \ b_1 &= \frac{1}{3}\left[11N - \frac{(N+3)(N^2-12)}{N-6}\right] \\ &= \frac{-N^3+8N^2-54N+36}{3(N-6)} \ . \end{aligned} \tag{5.89}$$

Evaluating this for the two solutions (5.85), we obtain $\bar{b}_1 = -4.695$ for $(N;k;1,p_3) = (9;3;1,10)$ and $\bar{b}_1 = -5.252$ for $(N;k;1,p_3) = (12;3;1,5)$. These are both negative, so neither of these theories is asymptotically free.

In a similar manner, we find that the anomaly-free $S_k\bar{A}_k$ theories with higher k are also not asymptotically free. Substituting the value $k = 4$ into Eq. (5.88) yields

$$\begin{aligned} k=4 \ \rightarrow \ b_1 &= \frac{1}{3}\left[11N - \frac{(N+3)(N+4)^2(N-4)}{3(N-8)}\right] \\ &= \frac{-N^4-7N^3+37N^2-152N+192}{9(N-8)} \ . \end{aligned} \tag{5.90}$$

Evaluating this for the solution $(N;4;1,p_4) = (10;4;1,39)$, we obtain $\bar{b}_1 = -64.67$. Finally, for $k = 5$,

$$k=5 \ \rightarrow b_1 \ = \ \frac{1}{3}\left[11N - \frac{(N+3)(N+4)(N+5)(N^2-20)}{12(N-10)}\right]$$

$$= \frac{-N^5 - 12N^4 - 27N^3 + 312N^2 - 380N + 1200}{36(N-10)} \quad (5.91)$$

Evaluating this for the solution $(N; 5; 1, p_5) = (11; 5; 1, 210)$, we get $\bar{b}_1 = -746.94$. For each of these theories, letting n_{cp} be larger than 1 makes b_1 more negative, so the respective theories with $n_{cp} \geq 2$ are also not asymptotically free.

Thus, we find that there are no anomaly-free chiral gauge theories with fermions in the k-fold symmetric and conjugate antisymmetric representations of SU(N) for $k \geq 3$.

5.5 Investigation of $S_k\bar{A}_\ell$ Chiral Gauge Theories with $k \neq \ell$ and $k,\ \ell \geq 2$

One can also consider generalizations of our $S\bar{A} = S_2\bar{A}_2$ chiral gauge theories to theories with chiral fermions transforming as the rank-k symmetric representation and the conjugate rank-ℓ antisymmetric representation of SU(N), where $k \neq \ell$ and $k,\ \ell \geq 2$. We consider theories of this type here. We denote the number of fermions transforming as the rank-k symmetric representation of SU(N) as n_{S_k} and the number of fermions transforming as the rank-ℓ conjugate antisymmetric representation as $n_{\bar{A}_\ell}$, respectively, and the theory itself as

$$(N; k; \ell; n_{S_k}, n_{\bar{A}_\ell}) \ . \quad (5.92)$$

Here the condition that there theory should have no anomaly in gauged currents reads

$$n_{S_k}\mathcal{A}(S_k) + n_{\bar{A}_\ell}\mathcal{A}(\bar{A}_\ell) = n_{S_k}\mathcal{A}(S_k) - n_{\bar{A}_\ell}\mathcal{A}(A_\ell) = 0 \ . \quad (5.93)$$

This anomaly cancellation condition is satisfied if and only if

$$\frac{n_{\bar{A}_\ell}}{n_{S_k}} = \frac{\mathcal{A}(S_k)}{\mathcal{A}(A_\ell)} \equiv p_{\bar{A}_\ell/S_k} = \frac{(N+k)!(N+2k)(N-\ell-1)!(\ell-1)!}{(N-3)!(N-2\ell)(N+2)!(k-1)!} \ . \quad (5.94)$$

Although $k \neq \ell$ here, we note that if one took $k = \ell$ as in the previous part of this chapter, then the correspondence in notation with Eqs. (5.5) and (5.79) is $p_{\bar{A}_k/S_k} \equiv p_k$ and $p_{\bar{A}_2/S_2} \equiv p_2 \equiv p$. For the ratio (5.94) to be a physical, positive number, it is necessary that

$$N \geq 2\ell + 1 \ , \quad (5.95)$$

and we shall restrict N to this range. As explicit examples, we discuss the $S_3\bar{A}_2$ and $S_2\bar{A}_3$ theories.

5.6 $S_3\bar{A}_2$ Theory

Here, in accordance with (5.95), we restrict N to the range $N \geq 5$. For this theory, the ratio $n_{\bar{A}_2}/n_{S_3}$ is

$$\frac{n_{\bar{A}_2}}{n_{S_3}} \equiv p_{\bar{A}_2/S_3} = \frac{(N+3)(N+6)}{2(N-4)} \ . \tag{5.96}$$

Unlike p_k for the case of diagonal $S_k\bar{A}_k$ theories, this ratio $p_{\bar{A}_2/S_3}$ is not a monotonic function of N. It decreases from the value 44 at $N = 5$ to a formal minimum at the real value $N = 4 + \sqrt{70} = 12.367$, where it is equal to $(17+2\sqrt{70})/2 = 16.667$, and then increases without bound as N increases further. Since $p_{\bar{A}_2/S_3}$ is always larger than unity, it is natural to consider models of this type with n_{S_3} equal to its smallest value, namely $n_{S_3} = 1$. We have found many of these, but none of them is asymptotically free. As allowed by the non-monotonicity of $p_{\bar{A}_2/S_3}$, there are two values of N that yield a minimal value of $p_{\bar{A}_2/S_3}$, namely $N = 11$ and $N = 14$, both of which give $p_{\bar{A}_2/S_3} = 17$. To minimize the fermion content with this value of $p_{\bar{A}_2/S_3}$, we choose $n_{S_3} = 1$ and $n_{\bar{A}_2} = 17$. For $N = 11$, we have $\mathcal{A}(S_3) = 119$ and $\mathcal{A}(\bar{A}_2) = -7$, while for $N = 14$, we have $\mathcal{A}(S_3) = 170$ and $\mathcal{A}(\bar{A}_2) = -10$. To test whether any of these solutions of the anomaly cancellation condition yields an asymptoticall free theory, we calculate the one-loop coefficient of the beta function. In general for this type of theory,

$$b_1 = \frac{1}{3}\left[11N - 2n_{S_3}\{T_{S_3} + p_{\bar{A}_2/S_3}T_{\bar{A}_2}\}\right] \ . \tag{5.97}$$

With the minimal choice $n_{S_3} = 1$, this is

$$b_1 = \frac{1}{3}\left[11N - \frac{(N+3)(N^2+N-10)}{N-4}\right] = \frac{-N^3+7N^2-37N+30}{3(N-4)} \ . \tag{5.98}$$

For the case with $N = 11$, $\bar{b}_1 = -3.263$, while for $N = 14$, $\bar{b}_1 = -4.934$. These are both negative, i.e., these theories are not asymptotically free. Solutions of the anomaly conditions with larger values of n_{S_3} and $n_{\bar{A}_2}$ yield values of b_1 that are even more negative. Thus, we do not find any anomaly-free, asymptotically free theories of this $S_3\bar{A}_2$ type.

5.7 $S_2\bar{A}_3$ Theories

5.7.1 General Analysis

Here we consider an SU(N) theory with n_{S_2} chiral fermions in the S_2 representation and $n_{\bar{A}_3}$ chiral fermions in the $\bar{A}_3$ representation. In accord with (5.95), we restrict N to the range $N \ge 7$. For this theory, the ratio $n_{\bar{A}_3}/n_{S_2}$ is

$$\frac{n_{\bar{A}_3}}{n_{S_2}} \equiv p_{\bar{A}_3/S_2} = \frac{2(N+4)}{(N-3)(N-6)} \ . \tag{5.99}$$

This ratio decreases monotonically as a function of N from the value value 11/2 at $N = 7$ and approaches zero as $N \to \infty$. The ratio (5.99) takes on an integer value for only one value of N, namely $N = 10$, where it is equal to 1. This reflects the equality $\mathcal{A}_{S_2} = 14 = \mathcal{A}_{A_3}$ for SU(10). A theory with $N = 10$ and $n_{cp} \ge 2$ copies of the S_2 and $\bar{A}_3$ representations is also anomaly-free.

For $N = 10$ and $n_{S_2} = n_{\bar{A}_3} = 1$, we calculate the reduced one-loop coefficient in the beta function to be $\bar{b}_1 = 1.8569$, so this theory satisfies the requirement of being asymptotically free. We compute the reduced two-loop coefficient to be $\bar{b}_2 = 0.086545$, so at the maximal scheme-independent level, i.e., the two-loop level, this theory has no IR zero in the beta function. Hence, as the reference scale μ decreases from the UV to the IR, the SU(10) gauge coupling continues to increase.

The condition of the cancellation of anomalies in gauged currents is also satisfied in a theory in which the chiral fermion content is replicated n_{cp} times. However, we find that only one of these nonminimal theories is asymptotically free, namely the one with $n_{cp} = 2$. For this theory with $N = 10$ and $n_{S_2} = n_{\bar{A}_3} = n_{cp} = 2$, we calculate $\bar{b}_1 = 0.795775$ and $\bar{b}_2 = -7.00383$. Thus, the two-loop beta function of this second theory has an IR zero at $\alpha_{IR,2\ell} = 0.1136$.

5.7.2 SU(10) Theory with $n_{S_2} = n_{\bar{A}_3} = 1$

Initial Breaking of SU(10) to SU(6)

We will focus here on the simplest SU(10) theory of this type, with $n_{cp} = 1$ and thus $n_{S_2} = n_{\bar{A}_3} = 1$. This theory has a classical global symmetry $G_{fl,cl} = \mathrm{U}(1)_{S_2} \otimes \mathrm{U}(1)_{\bar{A}_3}$. Both of these U(1) symmetries are broken by SU(10) instantons, but one can construct a linear combination $\mathrm{U}(1)'$ that is invariant in the presence of these instantons. Since (in a notation analogous to Eq. (5.22)) $\vec{v} = (T_{S_2}, T_{\bar{A}_3}) = (6, 14)$, $U(1)'$ has the charge assignments

$$(Q_{S_2}, Q_{\bar{A}_3}) \propto (7, -3) \ . \tag{5.100}$$

We have not found a set of gauge-singlet composite fermion operators satisfying the 't Hooft anomaly matching conditions for this $U(1)'$ symmetry. Therefore, we infer that as the SU(10) gauge coupling increases sufficiently, fermion condensation will occur. We find that the most attractive channel is

$$MAC: \quad \bar{A}_3 \times \bar{A}_3 \to A_4 \tag{5.101}$$

with

$$\Delta C_2 = 9.90 \quad \text{for } \bar{A}_3 \times \bar{A}_3 \to A_4 \ . \tag{5.102}$$

We denote the S_2 and $\bar{A}_3$ fermion fields as ψ_L^{ab} and $\chi_{abcd,L}$. The condensate for the channel $\bar{A}_3 \times \bar{A}_3 \to A_4$ is of the form

$$\langle \epsilon^{7\,8\,9\,10\,\{a_1 \ldots a_6\}} \chi^T_{a_1 a_2 a_3,L} C \chi_{a_4 a_5 a_6,L} \rangle \ , \tag{5.103}$$

where, by convention, we take the four uncontracted indices to be 7, 8, 9, and 10, and the summed indices to be $a_1, ..., a_6 \in \{1, ..., 6\}$. We denote the scale at which this condensate forms as Λ_{10}. This condensate breaks the SU(10) gauge symmetry to SU(6) and also breaks the global $U(1)'$ symmetry. The 64 gauge bosons in the coset SU(10)/SU(6) also gain masses of this order. In order to construct the low-energy effective SU(6) gauge theory that is operative at reference scales $\mu < \Lambda_{10}$, we first enumerate the chiral fermions that are involved in the condensate (5.103) and that consequently gain dynamical masses of order Λ_{10} and are integrated out to form this low-energy effective theory. The representation $\bar{A}_3$ has dimension $\binom{10}{3} = 120$ in SU(10). One can choose the three antisymmetrized group indices $a_1, \ a_2, \ a_3 \in \{1, ..., 6\}$ in the first fermion in (5.103) in any of $\binom{6}{3} = 20$ ways, and the remaining group indices $a_4, \ a_5, \ a_6$ in any of $\binom{3}{3} = 1$ ways, so of the initial 120 components in the $\bar{A}_3$ fermion, the 20 components with gauge indices in the set $\{1, ..., 6\}$ gain masses and are integrated out of the SU(6) theory.

We next must determine how the remaining massless fermions transform under SU(6). For this purpose, let us use group indices $a, b, .. \in \{1, ..., 6\}$ to refer to indices of the residual SU(6) gauge symmetry and $\alpha, \beta, .. \in \{7, \ 8, \ 9, \ 10\}$ to refer to the indices along the broken directions of SU(10). The remaining 100 massless components of the $\bar{A}_3$ fermion can be classified and enumerated as follows. First, there are the $\binom{4}{3} = 4$ components $\chi_{\alpha\beta\gamma,L}$

for which $\alpha,\ \beta,\ \gamma \in \{7,\ 8,\ 9,\ 10\}$, which are singlets under SU(6). Second, there are the $6 \times \binom{4}{2} = 36$ components $\chi_{a\alpha\beta,L}$ with $1 \le a \le 6$ and $7 \le \alpha,\ \beta \le 10$, which form six $\bar{F}$s of SU(6). Third, there are $\binom{6}{2} \times 4 = 60$ components $\chi_{ab\alpha,L}$ with $1 \le a,\ b \le 6$ and $7 \le \alpha \le 10$, which comprise four copies of $\bar{A}_2$ in SU(6). For the symmetric rank-2 tensor representation, we have

$$(S_2)_{\rm SU(10)} = (S_2)_{\rm SU(6)} + 4\,F_{\rm SU(6)} + 10\,(1)_{\rm SU(6)}\ , \tag{5.104}$$

where $(1)_{\rm SU(6)}$ is the singlet. Recall that $\dim(S_k) = (1/k!)\prod_{j=0}^{k-1}(N+j)$. Thus, the 55-dimensional $(S_2)_{\rm SU(10)}$ representation of SU(10) decomposes into the sum of the the $(S_2)_{\rm SU(6)}$ representation of SU(6) with its 21 component fields ψ_L^{ab} with $1 \le a, b \le 6$, plus four copies of the fundamental representation of SU(6) with fields $\psi_L^{a\alpha}$, $1 \le a \le 6$ and $7 \le \alpha \le 10$, and ten SU(6)-singlet fields $\psi_L^{\alpha\beta}$ with $7 \le \alpha,\ \beta \le 10$. We summarize the massless SU(6)-nonsinglet chiral fermion content of the low-energy SU(6) theory:

$$\text{SU}(6): \quad \text{fermions}:\ S_2 + 4\bar{A}_2 + 4\,F + 6\,\bar{F}\ . \tag{5.105}$$

The explicit fermion fields (with dimensionalities in parentheses) are

$$\begin{aligned}
&S_2(21): \quad \psi_L^{ab} \text{ with } 1 \le a,\ b \le 6 \\
&4\ \bar{A}_2(15): \quad \chi_{ab\alpha,L} \text{ with } 1 \le a,\ b \le 6 \text{ and } 7 \le \alpha \le 10 \\
&4\ F(6): \quad \psi^{a\alpha,L} \text{ with } 1 \le a \le 6 \text{ and } 7 \le \alpha \le 10 \\
&6\ \bar{F}(6): \quad \chi_{a\alpha\beta,L} \text{ with } 1 \le a \le 6 \text{ and } 7 \le \alpha,\ \beta \le 10\ .
\end{aligned} \tag{5.106}$$

As guaranteed by our theorem above, this low-energy effective SU(6) theory is anomaly-free; the contributions to the anomaly are

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}(S_2) - 4\mathcal{A}(A_2) + 4\mathcal{A}(F) + 6\mathcal{A}(\bar{F}) \\
&= 10 - (4 \times 2) + 4 - 6 = 0\ .
\end{aligned} \tag{5.107}$$

Breaking of SU(6) to SU(5)

We calculate the reduced one-loop and two-loop coefficients of the beta function of this SU(6) theory (5.105) to be $\bar{b}_1 = 0.84883$ and $\bar{b}_2 = -0.56465$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 1.503$. The most attractive channel for fermion condensation is $S_2 \times \bar{F} \to F$, with $\Delta C_2 = 20/3$ and the resultant estimate $\alpha_{cr} \simeq 0.31$. (The next-most attractive channel is $F \times \bar{F} \to 1$ with $\Delta C_2 = 35/6$.) The ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 4.8$, which is considerably larger than unity. We will explore evolution toward the infrared that involves further fermion condensation, breaking the SU(6) gauge symmetry to SU(5). We denote the scale at which such condensation occurs as Λ'_6 (where the prime is included to avoid confusion with the scale Λ_6 introduced in our discussion above of the SU(6) $S_2\bar{A}_2$ theory). By convention, we label the breaking direction as $a = 6$ and the α, β indices of the $\bar{F}$ fermion as $\alpha = 9,\ \beta = 10$. The associated $S_2\bar{F}$ fermion condensate is then

$$\langle \sum_{b=1}^{6} \psi_L^{6b\ T} C \chi_{b\alpha\beta,L} \rangle \quad \text{with } (\alpha,\beta) = (9,10) \ . \tag{5.108}$$

The fermions involved in this condensate gain dynamical masses of order Λ_6, as do the 11 gauge bosons in the coset SU(6)/SU(5).

Breaking of SU(5) to SU(4)

To analyze the subsequent evolution into the infrared, we enumerate the massless SU(5)-nonsinglet chiral fermion content of the resultant low-energy effective SU(5) theory. By the same methods as before, we find that this content is

$$\text{SU}(5) : \text{fermions} : \quad S_2 + 4\bar{A}_2 + 4F + 9\bar{F} \ . \tag{5.109}$$

The explicit fermion fields (with dimensionalities in parentheses) are listed below. For this purpose, we relabel the group indices such that $a, b \in \{1, ..., 5\}$ are SU(5) indices and $\alpha, \beta \in \{6, ..., 10\}$. We have

$$S_2(15) : \quad \psi_L^{ab} \text{ with } 1 \le a\ b \le 5$$

$$4\ \bar{A}_2(10) : \quad \chi_{ab\alpha,L} \text{ with } 1 \le a,\ b \le 5 \text{ and } 7 \le \alpha \le 10$$

$$4\ F(5) : \quad \psi_L^{a\alpha} \text{ with } 1 \le a \le 5 \text{ and } 7 \le \alpha \le 10$$

$$9\ \bar{F}(5) : \quad \chi_{a\alpha\beta,L} \text{ with } 1 \le a \le 5$$

$$\text{and } 6 \le \alpha,\ \beta \le 10 \text{ except } (\alpha,\beta) = (9,10) \ . \tag{5.110}$$

We calculate the reduced one-loop and two-loop coefficients of the beta function of this SU(5) theory to be $\bar{b}_1 = 0.61009$ and $\bar{b}_2 = -0.61384$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 0.994$. The most attractive channel for fermion condensation is $\bar{A}_2 \times \bar{A}_2 \to F$, with $\Delta C_2 = 24/5$ and the resultant estimate $\alpha_{cr} \simeq 0.44$. The resultant ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 2.3$, suggesting that this condensation could plausibly occur. With condensation in the $\bar{A}_2 \times \bar{A}_2 \to F$ channel, and with the breaking direction taken to be $a = 5$, the condensates are

$$\langle \epsilon^{abcd5} \chi^T_{ab\alpha,L} C \chi_{cd\beta,L} \rangle \propto \Big[\langle \chi^T_{12\alpha,L} C \chi_{34\beta,L} \rangle - \langle \chi^T_{13\alpha,L} C \chi_{24\beta,L} \rangle + \langle \chi^T_{14\alpha,L} C \chi_{23\beta,L} \rangle \Big] \ , \tag{5.111}$$

where $6 \le \alpha, \ \beta \le 10$ as specified above. We denote the scale at which these condensates form as Λ'_5. The fermions involved in these condensates, as well as the nine gauge bosons in the coset SU(5)/SU(4), gain masses of order Λ'_5.

IR Evolution of the Descendant SU(4) Theory

We determine the massless SU(4)-nonsinglet chiral fermion content of the resultant SU(4) descendant theory to be

$$\mathrm{SU}(4): \qquad \text{fermions}: \ S_2 + 5F + 13\bar{F} = S_2 + 8\bar{F} + 5\{F + \bar{F}\} \ . \tag{5.112}$$

We see that this is precisely the $N = 4$, $p = 5$ special case of the Sp model of Eq. (5.41) studied in [11, 22, 82]. For this theory we calculate the beta function coefficients $\bar{b}_1 = 0.5303$ and $\bar{b}_2 = -0.2496$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 2.125$. The fermion content of this theory satisfies the 't Hooft anomaly matching conditions [11, 22, 82], so one possibility is that as the gauge interaction becomes strong, the theory confines and produces massless composite SU(4)-singlet spin 1/2 fermions. Another possibility is that the gauge interaction produces fermion condensation. The most attractive channel is $S_2 \times \bar{F} \to F$ with $\Delta C_2 = 9/2$, so the rough estimate of α_{cr} is $\alpha_{cr} \simeq 0.42$. The resultant ratio $\rho = 5.1$ is well above unity, which renders it likely that either the gauge interaction confines and produces the above-mentioned massless composite fermions or it produces fermion condensation in this $S_2 \times \bar{F} \to F$ channel. These possibilities and the further evolution into the IR were discussed in detail in [22].

Chapter 6

Chiral Gauge Theories with Fermions in Fundamental and Antisymmetric Rank-2 Representations

In this chapter, we study asymptotically free chiral gauge theories with an SU(N) gauge group and chiral fermions transforming according to the antisymmetric rank-k tensor representation, $A_k \equiv [k]_N$, and the requisite number, $n_{\bar{F}}$, of copies of fermions in the conjugate fundamental representation, $\bar{F} \equiv \overline{[1]}_N$, to render the theories anomaly-free. We denote these as $A_k\,\bar{F}$ theories. We take $N \geq 2k+1$ so that $n_{\bar{F}} \geq 1$. The $A_2\,\bar{F}$ theories form an infinite family with $N \geq 5$, but we show that the $A_3\,\bar{F}$ and $A_4\,\bar{F}$ theories are only asymptotically free for N in the respective ranges $7 \leq N \leq 17$ and $9 \leq N \leq 11$, and that there are no asymptotically free $A_k\,\bar{F}$ theories with $k \geq 5$. We investigate the types of ultraviolet to infrared evolution for these $A_k\,\bar{F}$ theories and find that, depending on k and N, they may lead to a non-Abelian Coulomb phase, or may involve confinement with massless gauge-singlet composite fermions, bilinear fermion condensation with dynamical gauge and global symmetry breaking, or formation of multifermion condensates that preserve the gauge symmetry. We also show that there are no asymptotically free, anomaly-free SU(N) $S_k\,\bar{F}$ chiral gauge theories with $k \geq 3$, where S_k denotes the rank-k symmetric representation. The original result is published in [3].

6.1 $A_k\,\bar{F}$ Theories and Constraints from Anomaly Cancellation and Asymptotic Freedom

The chiral gauge theories that we study here have an SU(N) gauge group and chiral fermions transforming according to a rank-k antisymmetric tensor representation $A_k \equiv [k]_N$ of this group, and the requisite number of chiral fermions in the conjugate fundamental representation, $\bar{F} \equiv \overline{[1]}_N$, to render the theories free of any anomaly in gauged currents [107]. Here we determine the constraints on these theories from anomaly cancellation and asymptotic freedom. These theories are irreducibly chiral, i.e., they do not contain any vectorlike subsector. Consequently, the chiral gauge symmetry forbids any fermion mass terms in the underlying lagrangian. We denote the number of copies (flavors) of $\bar{F}$ fermions as $n_{\bar{F}}$. The contribution to the triangle anomaly in gauged currents of a chiral fermion in the A_k representation is [106] (see Appendix A)

$$\mathcal{A}([k]_N) = \frac{(N-3)!\,(N-2k)}{(N-k-1)!\,(k-1)!} \ . \tag{6.1}$$

The total anomaly in the theory is

$$\mathcal{A} = \mathcal{A}([k]_N) + n_{\bar{F}}\mathcal{A}(\overline{[1]}_N) = \mathcal{A}([k]_N) - n_{\bar{F}}\mathcal{A}([1]_N) \ , \tag{6.2}$$

so $\mathcal{A} = 0$, i.e., the theory is free of anomalies in gauged currents, if and only if

$$n_{\bar{F}} = \mathcal{A}([k]_N) \ . \tag{6.3}$$

If N is even and $k = N/2$, the $[k]_N = [k]_{2k}$ representation is self-conjugate, with zero anomaly, so Eq. (6.3) yields $n_{\bar{F}} = 0$ and a nonchiral theory. In order to get a chiral theory, with positive $n_{\bar{F}}$, it is necessary and sufficient that

$$N \geq N_{min} = 2k+1 \ , \tag{6.4}$$

so, for a given k, we will restrict N to this range. For N in this range, the anomaly $\mathcal{A}([k]_N)$ is an integer greater than unity. A member of this set of chiral gauge theories is thus determined by its values of k and N and has the form

$$G = \mathrm{SU}(N), \ \text{fermions}: \ A_k + n_{\bar{F}}\,\bar{F} \ , \tag{6.5}$$

i.e., $[k]_N + n_{\bar{F}}\overline{[1]}_N$, where N is bounded below by (6.4). The $A_k\,\bar{F}$ theories with $k = 3$ and $k = 4$ have respective upper bounds on N imposed by the requirement of asymptotic freedom.

To determine the upper bounds on N for these values of k, we calculate the one-loop coefficient in the beta function, b_1. To indicate explicitly the dependence of the b_ℓ coefficients with $\ell = 1, 2$ on k, we shall write them as $b_\ell^{(k)}$. For a general SU(N) $A_k\,\bar{F}$ theory, we have

$$\begin{aligned} b_1^{(k)} &= \frac{1}{3}\Big[11N - 2\Big\{T(A_k) + n_{\bar{F}}\,T(\bar{F})\Big\}\Big] \\ &= \frac{1}{3}\Big[11N - \frac{1}{(k-1)!}\Big\{\Big[\prod_{j=2}^{k}(N-j)\Big] \\ &+ \frac{(N-3)!(N-2k)}{(N-k-1)!}\Big\}\Big] \ . \end{aligned} \tag{6.6}$$

In Eq. (6.6), both $T(A_k)$ and $n_{\bar{F}} = \mathcal{A}(A_k)$ are polynomials of degree $\max(1, k-1)$ in N and hence, $b_1^{(k)}$ is a polynomial of degree $\max(1, k-1)$ in N. Specifically, we find

$$b_1^{(2)} = 3N + 2 \ , \tag{6.7}$$

$$b_1^{(3)} = \frac{1}{3}(-N^2 + 18N - 12) \ , \tag{6.8}$$

and

$$b_1^{(4)} = \frac{1}{9}(-N^3 + 12N^2 - 14N + 60) \ . \tag{6.9}$$

The $A_2\,\bar{F}$ theories are thus asymptotically free without any upper bound on N. For the $A_3\,\bar{F}$ theories, the asymptotic-freedom requirement that $b_1^{(3)}$ must be positive yields the upper bound $N \le 17$. (If one were to generalize N from positive integer values to real values, $b_1^{(3)}$ is positive for N in the range $9 - \sqrt{69} < N < 9 + \sqrt{69}$, i.e., $0.6934 < N < 17.3066$ to the indicated floating-point accuracy.) In the $A_4\,\bar{F}$ theories, $b_1^{(4)}$ is positive only for the integer values $N = 9,\ 10,\ 11$. (With N generalized to a positive real number, $b_1^{(4)} > 0$ for $N < 11.2291$.) Denoting N_{max} as the maximal value of N, for a given k, for which an SU(N) $A_k\bar{F}$ theory is asymptotically free, we summarize these results as

$$N_{max} = \begin{cases} \infty & \text{for } k = 2 \\ 17 & \text{for } k = 3 \\ 11 & \text{for } k = 4 \end{cases} \ . \tag{6.10}$$

Combining these results, we explicitly exhibit the asymptotically free, anomaly-free chiral gauge theories of this type with $2 \le k \le 4$ together with the respective allowed ranges of N, $N_{min} \le N \le N_{max}$:

$$k = 2 \implies N \ge 5 \ , \tag{6.11}$$

$$k = 3 \implies 7 \le N \le 17 \ , \tag{6.12}$$

$$k = 4 \implies 9 \le N \le 11 \ . \tag{6.13}$$

The SU(N) $A_2\,\bar{F}$ theories have been studied in several works [9,11,18,22, 82,83,108], has fermion content given by

$$k = 2: \quad \text{fermions}: A_2 + (N-4)\,\bar{F} \ , \tag{6.14}$$

The SU(N) $A_k\,\bar{F}$ theories with $k = 3$ and $k = 4$ are, to our knowledge, new here. These have the fermion contents

$$k = 3: \qquad \text{fermions}: A_3 \ + \ \frac{(N-3)(N-6)}{2}\,\bar{F} \tag{6.15}$$

and

$$k = 4: \qquad \text{fermions}: A_4 \ + \ \frac{(N-3)(N-4)(N-8)}{6}\,\bar{F} \ . \tag{6.16}$$

For the $A_k\,\bar{F}$ theories with $k = 2,\ 3,\ 4$, we denote the fermion field in the $A_k = [k]_N$ representation as ψ_L^{ab}, ψ_L^{abd}, and ψ_L^{abde}, respectively, where a, b, d, e are SU(N) gauge indices (the symbol c is reserved to mean charge conjugation) with N is in the respective intervals (6.14)-(6.16), and we denote the $\bar{F}$ fermions as $\chi_{a,i,L}$, where i is a copy (flavor) index taking values in the respective ranges $1 \le i \le n_{\bar{F}} = \mathcal{A}([k]_N)$.

We next show that there are no asymptotically free $A_k\,\bar{F}$ theories with $k \ge 5$. Consider first the $k = 5$ theory, for which

$$b_1^{(5)} = \frac{1}{36}(-N^4 + 18N^3 - 119N^2 + 474N - 360) \ . \tag{6.17}$$

With N generalized to a real variable, $b_1^{(5)}$ is positive only for N in the range $0.9585 < N < 10.7379$. But for an $A_k\,\bar{F}$ theory, N is bounded below by $2k+1$, which has the value 11 here, so for this $k = 5$ theory there is no value of N that simultaneously satisfies both the lower bound (6.4) and the requirement of asymptotic freedom. We reach the same conclusion in the $k = 6$ case, for which

$$b_1^{(6)} = \frac{1}{180}(-N^5 + 25N^4 - 245N^3 + 1175N^2 - 2094N + 2520) \ . \tag{6.18}$$

With N extended from physical values to real numbers, $b_1^{(6)}$ is positive if $N < 11.098$, but N is required to satisfy $N \geq 13$, which again means that for $k = 6$ there is no value of N that satisfies the lower bound (6.4) and the requirement of asymptotic freedom. Similarly, we find that for the $k = 7$ case, $b_1^{(7)}$ is only positive for the range $1.094 < N < 11.742$, while N must be in the range $N \geq 15$ by (6.4), and so forth for higher k. The underlying reason for the non-existence of asymptotically free $A_k \bar{F}$ chiral gauge theories with these higher values of k is that, as noted above, both $T([k]_N)$ and $\mathcal{A}([k]_N)$ are polynomials of degree $\max(1, k-1)$ in N, and they both contribute negatively to $b_1^{(k)}$ for the relevant range $N \geq 2k + 1$. Their negative contributions eventually outweigh the positive contribution of the $(11/3)N$ term from the gauge fields.

In passing, we remark that there are two possible ways that one could expand the fermion content of the $A_k \bar{F}$ models considered here for certain k and N values, as restricted by the constraint of asyamptotic freedom, namely (i) to have n_{cp} replications of the chiral fermion content and (ii) to add vectorlike subsectors. For example, in category (i), the following $k = 3$ theories are asymptotically free: $n_{cp} = 2$ and $7 \leq N \leq 11$; $n_{cp} = 3$ and $7 \leq N \leq 9$; $n_{cp} = 4$ and $N = 7,\ 8$; and $n_{cp} = 5$ and $N = 7$. We have studied different chiral gauge theories with this sort of n_{cp} replication of a minimal irreducible chiral fermion content in [2]. We shall not pursue these expansions here but instead focus on studying the minimal $A_k \bar{F}$ theories.

6.2 Beta Function Analysis of $A_k \bar{F}$ Theories

In this section we give a general analysis of the beta function applicable to all of the (anomaly-free) asymptotically free $A_k \bar{F}$ theories, with N in the respective ranges $N \geq 5$ for $k = 2$ and the finite intervals $7 \leq N \leq 17$ for $k = 3$ and $9 \leq N \leq 11$ for $k = 4$ as given in (6.14)-(6.16). In Sect. 6.1 we gave the one-loop coefficient for the $A_k \bar{F}$ theories, which we used to determine the upper bound on N for a given k. Here we proceed to give the two-loop coefficient, $b_2^{(k)}$, and use it to analyze the UV to IR evolution. We have (again with $A_k \equiv [k]_N$, and $F \equiv [1]_N$)

$$b_2^{(k)} = \frac{1}{3}\Big[34N^2 - 2\Big\{\Big(5C_2(G) + 3C_2(A_k)\Big)T(A_k)$$

$$+ \quad n_{\bar{F}}\Big(5C_2(G) + 3C_2(\bar{F})\Big)T(\bar{F})\Big\}\Bigg] \ , \tag{6.19}$$

where the various group invariants are listed in appendix A. For the three relevant cases, $k = 2,\ 3,\ 4$, the explicit expressions are

$$b_2^{(2)} = \frac{13N^3 + 30N^2 + N - 12}{2N} \ , \tag{6.20}$$

$$b_2^{(3)} = \frac{-16N^4 + 183N^3 - 204N^2 - 27N + 108}{6N} \ , \tag{6.21}$$

and

$$\begin{aligned} b_2^{(4)} &= (36N)^{-1}\Big[- 35N^5 + 429N^4 - 1321N^3 + 2235N^2 \\ &+ 588N - 1440\Big] \ . \end{aligned} \tag{6.22}$$

In Table 6.1 we list values of the reduced coefficients $\bar{b}_1$ and $\bar{b}_2$ for an illustrative set of the $A_2\,\bar{F}$ theories and for all of the (asymptotically free) $A_3\,\bar{F}$ and $A_4\,\bar{F}$ theories. In the cases where $\bar{b}_2 < 0$ so that the two-loop beta function has a physical IR zero, we have also listed the value of $\alpha_{IR,2\ell}$. The value of the resultant ratio ρ_c for condensation in the most attractive channel for bilinear fermion condensation (discussed further below) gives an estimate of whether the theories are weakly or strongly coupled in the infrared. This is indicated by the abbreviations WC, MC, and SC (weak coupling, moderate coupling, and strong coupling) in Table 6.1.

6.3 Global Symmetry of $A_k\,\bar{F}$ Theories

Because the $A_k\,\bar{F}$ theories are irreducibly chiral, so that the chiral gauge symmetry requires the fermions to be massless, each such theory has a classical global flavor symmetry

$$G_{fl,cl}^{(k)} = {\rm U}(n_{\bar{F}})_{\bar{F}} \otimes {\rm U}(1)_{A_k} \ , \tag{6.23}$$

where $n_{\bar{F}} = \mathcal{A}([k]_N)$ as given in Eq. (6.3). Equivalently,

$$G_{fl,cl}^{(k)} = \begin{cases} {\rm U}(1)_{\bar{F}} \otimes {\rm U}(1)_{A_k} & \text{if } n_{\bar{F}} = 1 \\ {\rm SU}(n_{\bar{F}})_{\bar{F}} \otimes {\rm U}(1)_{\bar{F}} \otimes {\rm U}(1)_{A_k} & \text{if } n_{\bar{F}} \ge 2 \end{cases} \ . \tag{6.24}$$

Table 6.1: Some properties of SU(N) $A_k\,\bar{F}$ chiral gauge theories. The quantities listed are k, N, $n_{\bar{F}}$, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the most attractive bilinear fermion condensation channel in the SU(N) theory, and the ratio ρ_c. The dash notation $-$ means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, with the abbreviations SC, MC, WC for the type coupling in the IR (SC = strong, MC = moderate, WC = weak coupling). In the WC case, the UV to IR evolution is to a non-Abelian Coulomb phase (NACP). The various possibilities for the evolution involving strong and moderately strong coupling are discussed in the text. For $k = 2$, we include illustrative results covering the interval $5 \le N \le 10$; for $k = 3,\ 4$ we list results for all (asymptotically free) $A_k\,\bar{F}$ theories.

k	N	$n_{\bar{F}}$	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{IR,2\ell}$	α_{cr}	ρ_c	IR coupling
2	5	1	1.3528	1.4996	$-$	0.44	$-$	SC
2	6	2	1.59155	2.0486	$-$	0.45	$-$	SC
2	7	3	1.8303	2.6796	$-$	0.37	$-$	SC
2	8	4	2.0690	3.3927	$-$	0.31	$-$	SC
2	9	5	2.3077	4.1879	$-$	0.27	$-$	SC
2	10	6	2.5465	5.0654	$-$	0.24	$-$	SC
3	7	2	1.7242	2.1525	$-$	0.20	$-$	SC
3	8	5	1.8038	1.9784	$-$	0.21	$-$	SC
3	9	9	1.8303	1.3805	$-$	0.21	$-$	SC
3	10	14	1.8038	0.2573	$-$	0.21	$-$	SC
3	11	20	1.7242	-1.4926	1.155	0.21	5.4	SC
3	12	27	1.59155	-3.9705	0.4008	0.21	1.9	MC
3	13	35	1.4059	-7.2779	0.1932	0.19	0.99	MC
3	14	44	1.1671	-11.5161	0.1013	0.18	0.57	MC
3	15	54	0.8753	-16.7864	0.05215	0.16	0.32	WC, NACP
3	16	65	0.5305	-23.1901	0.02288	0.15	0.15	WC, NACP
3	17	77	0.1326	-30.8287	0.00430	0.14	0.03	WC, NACP
4	9	5	1.5650	-0.5896	2.6542	0.12	22.5	SC
4	10	14	1.0610	-5.3310	0.1990	0.12	1.7	MC
4	11	28	0.2387	-13.410	0.0178	0.12	0.15	WC, NACP

For $n_{\bar{F}} \geq 2$, the multiplet $(\chi_{a,1,L}, ..., \chi_{a,n_{\bar{F}},L})$ may be taken to transform as the conjugate fundamental, $\overline{\square}$, representation of the global flavor group, $\mathrm{SU}(n_{\bar{F}})$. The $\mathrm{U}(1)_{\bar{F}}$ and $\mathrm{U}(1)_{A_k}$ symmetries in (6.24) are both broken by $\mathrm{SU}(N)$ instantons [6]. As in [22], we define a vector whose components are comprised of the instanton-generated contributions to the breaking of these symmetries. In the basis $(A_k, \bar{F})$, this vector is

$$\vec{v}^{(k)} = \Big(T([k]_N),\ n_{\bar{F}}T(\bar{F})\Big) = \lambda_{N,k}(N-2, N-2k)\ , \tag{6.25}$$

where

$$\lambda_{N,k} = \frac{(N-3)!}{2(k-1)!(N-k-1)!}\ . \tag{6.26}$$

We can construct one linear combination of the two original currents that is conserved in the presence of $\mathrm{SU}(N)$ instantons. We denote the corresponding global $\mathrm{U}(1)$ flavor symmetry as $\mathrm{U}(1)'$ and the fermion charges under this $\mathrm{U}(1)'$ as

$$\vec{Q}^{(k)\prime} = \Big(Q'_{A_k},\ Q'_{\bar{F}}\Big)\ . \tag{6.27}$$

The $\mathrm{U}(1)'$ current is conserved if and only if

$$\sum_f n_f T(R_f)\, Q_f^{(k)\prime} = \vec{v}\cdot\vec{Q}^{(k)\prime} = 0\ . \tag{6.28}$$

This condition only determines the vector $\vec{Q}^{(k)\prime}$ up to an overall multiplicative constant. A solution is

$$\vec{Q}^{(k)\prime} = \Big(N-2k,\ -(N-2)\Big)\ . \tag{6.29}$$

The actual global chiral flavor symmetry group (preserved in the presence of instantons) is then

$$G_{fl}^{(k)} = \begin{cases} \mathrm{U}(1)' & \text{if } n_{\bar{F}} = 1 \\ \mathrm{SU}(n_{\bar{F}})\otimes \mathrm{U}(1)' & \text{if } n_{\bar{F}} \geq 2 \end{cases}\ . \tag{6.30}$$

For the three k values relevant here, this is

$$G_{fl}^{(2)} = \begin{cases} \mathrm{U}(1)' & \text{if } N = 5 \\ \mathrm{SU}(N-4)_{\bar{F}}\otimes \mathrm{U}(1)' & \text{if } N \geq 6 \end{cases}\ , \tag{6.31}$$

with $\mathrm{U}(1)'$ charges

$$\vec{Q}^{(2)\prime} = \Big(N-4,\ -(N-2)\Big)\ , \tag{6.32}$$

$$G_{fl}^{(3)} = \mathrm{SU}\Big(\frac{(N-3)(N-6)}{2}\Big)_{\bar F} \otimes \mathrm{U}(1)' \tag{6.33}$$

with $\mathrm{U}(1)'$ charges

$$\vec{Q}^{(3)\prime} = \Big(N-6,\ -(N-2)\Big)\ , \tag{6.34}$$

and

$$G_{fl}^{(4)} = \mathrm{SU}\Big(\frac{(N-3)(N-4)(N-8)}{6}\Big)_{\bar F} \otimes \mathrm{U}(1)' \tag{6.35}$$

with $\mathrm{U}(1)'$ charges

$$\vec{Q}^{(4)\prime} = \Big(N-8,\ -(N-2)\Big)\ . \tag{6.36}$$

6.4 Most Attractive Channel for Bilinear Fermion Condensation in $A_k\,\bar F$ Theories

6.4.1 General Analysis

The ultraviolet to infrared evolution of a particular $\mathrm{SU}(N)$ $A_k\,\bar F$ theory is determined by the values of N and k. In the cases where it can lead to the formation of a bilinear fermion condensate, one should then determine the most attractive channel in which this condensate can form. We present this analysis here. Since the $A_k\,\bar F$ theories that we consider here are irreducibly chiral, a bilinear condensate breaks the gauge symmetry. In Sect. 6.8 below, we will discuss the possible formation of multifermion condensates involving more than just two fermions, which can preserve the chiral gauge symmetry.

For the theories that we are discussing here, there are two relevant bilinear fermion condensation channels. First, there is a channel with a condensate that involves the contraction of $2k$ gauge indices of the antisymmetric tensor density $\epsilon_{a_1,\ldots,a_N}$ with the bilinear fermion product $A_k \times A_k$, which transforms like $\bar A_{N-2k}$. This channel can thus be written as

$$A_k \times A_k \to \bar A_{N-2k}\ . \tag{6.37}$$

This channel has attractiveness measure

$$\Delta C_2 = \frac{k^2(N+1)}{N} \quad \text{for } A_k \times A_k \to \bar A_{N-2k}\ . \tag{6.38}$$

For a given k, this ΔC_2 is a monotonically decreasing function of N, decreasing gradually from its value at $N = 2k+1$,

$$\Delta C_2 = \frac{2k^2(k+1)}{2k+1} = \frac{(N-1)^2(N+1)}{4N} \quad \text{at } N = 2k+1$$
$$\text{for } A_k \times A_k \to \bar{A}_{N-2k} = \bar{A}_1 = \bar{F} \tag{6.39}$$

and approaching the limit k^2 for $N \gg k$.

Second, there is the channel

$$A_k \times \bar{F} \to A_{k-1} \ , \tag{6.40}$$

with

$$\Delta C_2 = \frac{(N+1)(N-k)}{N} \quad \text{for } A_k \times \bar{F} \to A_{k-1} \ . \tag{6.41}$$

For a given k, this ΔC_2 is a monotonically increasing function of N, increasing from the value

$$\Delta C_2 = \frac{2(k+1)^2}{2k+1} = \frac{(N+1)^2}{2N} \quad \text{at } N = 2k+1$$
$$\text{for } A_k \times \bar{F} \to A_{k-1} \ , \tag{6.42}$$

and approaching a linear growth with N for $N \gg k$. In Table 6.2 we list the value of ΔC_2 in Eq. (6.38) for the $A_k \times A_k \to \bar{A}_{N-2k}$ channel and the value of ΔC_2 in Eq. (6.41) for the $A_k \times \bar{F} \to A_{k-1}$ channel for an illustrative set of $A_2\,\bar{F}$ theories and for the full set of (asymptotically free) $A_3\,\bar{F}$ and $A_4\,\bar{F}$ theories.

The most attractive channel for bilinear fermion condensation is the one among these two channels with the larger value of ΔC_2 (assuming that these two values are unequal; we discuss the cases where they are equal below). For a given value of k, we thus determine the MAC as a function of N in its allowed range $N_{min} \le N \le N_{max}$ by examining the difference,

$$\Delta C_2(A_k \times A_k \to \bar{A}_{N-2k}) - \Delta C_2(A_k \times \bar{F} \to A_{k-1}) = \Big(\frac{N+1}{N}\Big)\Big[k(k+1) - N\Big] \ . \tag{6.43}$$

For $N = N_{min} = 2k+1$, ΔC_2 is larger for the first channel, $A_k \times A_k \to \bar{A}_{N-2k} = \bar{A}_1 = \bar{F}$, than for the second channel, $A_k \times \bar{F} \to A_{k-1}$. This is evident analytically from the fact that with $N = 2k+1$, the difference (6.43) is

$$\frac{2(k+1)(k^2-k-1)}{2k+1} = \frac{(N+1)(N^2-4N-1)}{4N} \ , \tag{6.44}$$

which is positive for the relevant range $k \geq 2$ considered here. Since ΔC_2 for the first channel decreases monotonically as a function of N, while the ΔC_2 for the second channel increases monotonically as a function of N, it follows that at some value of N, which we denote N_e (where e stands for "equal"), these values are equal, and for $N > N_e$, the ΔC_2 for the second channel is larger than that for the first channel. Setting the two ΔC_2 values equal and solving for $N = N_e$, we find

$$N_e = k(k+1) \ . \tag{6.45}$$

Evaluating Eq. (6.45) for the three relevant values of k, we have

$$N_e = \begin{cases} 6 & \text{for k=2} \\ 12 & \text{for k=3} \\ 20 & \text{for k=4} \end{cases} \ . \tag{6.46}$$

The first two of these values are within the respective allowed ranges for N, while the value for $k = 4$ is larger than the upper bound $N_{max} = 11$ for $k = 4$.

Consequently, with $N_{min} = 2k + 1$ and N_{max} as given in Eqs. (6.4) and (6.10), we find that, for a given k,

$$\begin{aligned} &\text{If } 2k+1 \leq N < k(k+1) \text{ then} \\ &\text{MAC} \ = \ A_k \times A_k \to \bar{A}_{N-2k} \ , \\ &\text{If } k(k+1) < N \leq N_{max} \text{ then} \\ &\text{MAC} \ = \ A_k \times \bar{F} \to A_{k-1} \ , \end{aligned} \tag{6.47}$$

with the proviso that the second possibility only applies if $k(k+1) < N_{max}$, and hence only for $k = 2$ and $k = 3$. Thus, in particular, if $N = N_{min} = 2k+1$, then the MAC is the special case of (6.37):

$$\text{If } N = 2k+1 \text{ then MAC} = A_k \times A_k \to \bar{F} \ . \tag{6.48}$$

In addition to breaking the original SU(N) gauge symmetry, these condensates also break both the non-Abelian factor group SU($n_{\bar{F}}$) (which is present if $n_{\bar{F}} \geq 2$) and the U(1)$'$ factor group in the global flavor symmetry (6.30). In particular, the breaking of the U(1)$'$ symmetry is evident from the fact that the respective condensates in these channels have the nonzero U(1)$'$ charges

$$Q'^{(k)} = 2Q'_{A_k} = 2(N-2k) \quad \text{for } A_k \times A_k \to \bar{A}_{N-2k} \tag{6.49}$$

and

$$Q'^{(k)} = Q'_{A_k} + Q'_{\bar{F}} = 2(1-k) \quad \text{for } A_k \times \bar{F} \to A_{k-1} \ . \tag{6.50}$$

The marginal case $N = N_e = k(k+1)$ requires further analysis, since the ΔC_2 values for the $A_k \times A_k \to \bar{A}_{N-2k}$ and $A_k \times \bar{F} \to A_{k-1}$ channels are equal, so the procedure of picking the channel with the largest ΔC_2 cannot determine which is more likely to occur. To deal with this marginal case, we use a vacuum alignment argument, which, as applied to possible bilinear fermion condensation channels, favors the one whose condensate respects the larger residual gauge symmetry. To apply the vacuum alignment argument, we must thus determine the residual gauge symmetry group respected by the condensates that occur in these two channels. The resultant bilinear fermion condensate transforms like an n-fold antisymmetric tensor representation of SU(N), where $n = N - 2k$ for the $A_k \times A_k \to \bar{A}_{N-2k}$ channel and $n = k - 1$ for the $A_k \times \bar{F} \to A_{k-1}$ channel. (The fact that in the first case the condensate transforms like $\bar{A}_{N-2k}$ rather than A_{N-2k} does not affect how this breaks SU(N).) From the point of view of the group theory, the problem of determining the residual gauge symmetry is effectively the same as the problem of determining the residual gauge symmetry that results when one has a Higgs field transforming according to the antisymmetric rank-n representation of SU(N). An analysis of this, within the context of Higgs-induced symmetry breaking, was given in [109], and the results depend, in that context, on the parameters in the Higgs potential, which one has the freedom to choose, subject to the overall constraint that the energy must be bounded below. As emphasized in Ref. [27], the situation is different in dynamical gauge symmetry breaking; in principle, given an initial gauge group and set of fermions, there is a unique answer for how the symmetry breaks; this breaking does not depend on any parameters in a Higgs potential. Despite this basic difference between dynamical and Higgs-induced gauge symmetry breaking, we can make use of the general group-theoretic analysis performed for the Higgs case. The result is that there are, *a priori*, three possibilities for the gauge symmetries respected by a condensate or Higgs vacuum expectation value transforming as the rank-n antisymmetric tensor representation of SU(N), $[n]_N$. Denoting the integral part of a real number r as $[r]$ and setting

$$\kappa \equiv [N/n] \ , \tag{6.51}$$

these are [109]

$$\mathrm{SU}(N-n) \otimes \mathrm{SU}(n) \quad \text{with } 2 \le n < [N/2] \ , \tag{6.52}$$

$$[\mathrm{SU}(n)]^{\kappa} \quad \text{with } 3 \le n \le [N/2] \text{ if } N - [N/n]n = 0 \text{ or } 1 \ , \tag{6.53}$$

and the symplectic group

$$\mathrm{Sp}(2\kappa) \quad \text{if } n = 2 \ . \tag{6.54}$$

We analyze the respective cases $k = 2,\ 3,\ 4$ next.

6.4.2 Case $k = 2$

From the special case for $k = 2$ of our general result (6.47) above, we infer that the $A_2 \times A_2 \to \bar{A}_1 = \bar{F}$ channel is the most attractive channel for bilinear fermion condensation in the $A_2\bar{F}$ theories for the lowest value of N, namely $N = 5$, while the $A_2 \times \bar{F} \to F$ channel is the MAC for the infinite interval $N \ge 7$. For the marginal case $k = 2$, $N = 6$, the $A_2 \times A_2 \to \bar{A}_{N-2k} = \bar{A}_2$ and $A_2 \times \bar{F} \to F$ channels have the same value of ΔC_2, namely $\Delta C_2 = 14/3 = 4.667$ (see Table 6.2), so the ΔC_2 attractiveness criterion cannot be used to decide which is more likely to occur. Now the condensate in the $A_2 \times \bar{F} \to F$ channel leaves invariant an SU(5) subgroup of SU(6), with order 24. To analyze the possible invariance groups of a condensate in the $A_2 \times A_2 \to \bar{A}_2$ channel, we apply our discussion above with $N = 6$, $n = 2$, and hence $\kappa = [6/2] = 3$, so the *a priori* possible invariance groups of the condensate are $\mathrm{SU}(4) \otimes \mathrm{SU}(2)$ with order 18 and Sp(6) with order 21. Neither of these groups has an order as large as that of SU(5), so the vacuum alignment argument predicts that, if a bilinear fermion condensate forms, then this condensate will form in the $A_2 \times \bar{F} \to F$ channel. Summarizing our results for $k = 2$ and all N, we thus find that if bilinear fermion condensation occurs, then

$$k = 2 \implies \mathrm{MAC} = \begin{cases} A_2 \times A_2 \to \bar{A}_1 & \text{for } N = 5 \\ A_2 \times \bar{F} \to F & \text{for } N \ge 6 \end{cases} \ . \tag{6.55}$$

As noted above, since this class of (asymptotically free) $A_2\,\bar{F}$ chiral gauge theories satisfies the 't Hooft global anomaly matching conditions, there is also the possibility of confinement, yielding massless composite fermions. There is also the possibility of multifermion condensate formation, which we will discuss below. Since the early works such as [8, 9, 11, 18], for a class of asymptotically free chiral gauge theories such as the $A_2\,\bar{F}$ class discussed here, for which the UV to IR evolution leads to strong coupling and hence could lead to confinement with massless composite fermions or to fermion condensation, there has not, to our knowledge, been a rigorous argument presented that actually determines the type of UV to IR evolution in an asymptotically free chiral gauge theory.

6.4.3 Case $k = 3$

From the special case for $k = 3$ of our general result (6.47) above, we infer that the $A_3 \times A_3 \to \bar{A}_{N-2k} = \bar{A}_{N-6}$ channel is the most attractive channel for bilinear fermion condensation not only for the minimal value of N, namely $N = 7$, but also for the interval of N values up to $N = 11$. We discuss the marginal case of $N = 12$ last. Again substituting $k = 3$ into (6.47), it follows formally that the MAC for $12 \le N \le 17$ is the $A_3 \times \bar{F} \to A_2$ channel. However, for $N = 13,\ 14$, the respective values of the IR zero in the beta function are sufficiently close to the rough estimate of the minimal critical value of α for condensate formation in the $A_3 \times \bar{F} \to A_2$ channel (see Tables 6.1 and 6.2) that it is possible that the system could evolve from the UV to a deconfined, non-Abelian Coulomb phase in the IR with no fermion condensate formation or associated spontaneous chiral symmetry breaking.

The $k = 3$, $N = 12$ case is again marginal; the $A_3 \times A_3 \to \bar{A}_6$ and $A_3 \times \bar{F} \to A_2$ channels have the same value of ΔC_2, namely $\Delta C_2 = 39/4 = 9.750$. Hence, we use a vacuum alignment argument to decide on which of these channels is more likely to occur. For the $A_3 \times A_3 \to \bar{A}_{N-6} = \bar{A}_6$ channel, we apply our discussion above with $N = 12$, $n = 6$, and hence $\kappa = [12/6] = 2$, so the invariance group of the $\bar{A}_2$ condensate is $[\mathrm{SU}(6)]^2$, with order 70. For the $A_3 \times \bar{F} \to A_2$ channel, we have $N = 12$, $n = 2$ and hence $\kappa = [12/2] = 6$, so the *a priori* possible invariance groups of the A_2 condensate are $\mathrm{SU}(10) \otimes \mathrm{SU}(2)$ with order 102 and $\mathrm{Sp}(12)$ with order 78. The vacuum alignment argument thus favors condensation in the $A_3 \times \bar{F} \to \bar{A}_2$

channel for this $N = 12$ case. Summarizing these results, we have

$$k = 3 \implies \text{MAC} = \begin{cases} A_3 \times A_3 \to \bar{A}_{N-6} & \text{for } 7 \le N \le 11 \\ A_3 \times \bar{F} \to A_2 & \text{for } 12 \le N \le 17 \end{cases} . \quad (6.56)$$

However, as mentioned above, for $N = 13,\ 14$ (and also for $N = 12$), the respective values of ρ_c are sufficiently close to unity that, in view of the intrinsic theoretical uncertainties in the analysis of the strong-coupling physics, it is possible that the UV to IR evolution could lead either to the formation of a fermion condensate or to a non-Abelian Coulomb phase without spontaneous chiral symmetry breaking.

If N is in the higher interval $15 \le N \le 17$, then ρ_c is sufficiently small that we definitely expect the evolution to lead to a chirally symmetric non-Abelian Coulomb phase in the IR. Hence, in these cases, the MAC is not directly relevant to the dynamics of the theory.

6.4.4 Case $k = 4$

Finally, we discuss the theories with $k = 4$, for which the interval of values of N is $9 \le N \le 11$. Since the value of N_e, namely $N_e = 20$, is larger than N_{max}, the most attractive channel for bilinear fermion condensation in all of these theories is $A_4 \times A_4 \to \bar{A}_{N-8}$, i.e., $A_4 \times A_4 \to \bar{F}$ for $N = 9$, $A_4 \times A_4 \to \bar{A}_2$ for $N = 10$, and $A_4 \times A_4 \to \bar{A}_3$ for $N = 11$. In the SU(9) $A_4\,\bar{F}$ theory, the IR zero in the two-loop beta function is much larger than α_{cr} for this channel, so it is likely that the SU(9) gauge interaction would produce a condensate in this channel, thereby breaking SU(9) to SU(8). For $N = 10$, $\alpha_{IR,2\ell}/\alpha_{cr} = 1.7$, which is sufficiently close to unity that, taking account of the uncertainties in the strong-coupling estimates, the UV to IR evolution might produce a condensate in the respective most attractive bilinear fermion channel or might lead to a non-Abelian Coulomb phase. For $N = 11$, the IR zero in the two-loop beta function is small compared with the estimated α_{cr} for the $A_4 \times A_4 \to \bar{A}_3$ condensation channel, so we definitely expect the system to evolve from the UV to a non-Abelian Coulomb phase in the IR.

Table 6.2: ΔC_2 values for the SU(N) $A_k\,\bar{F}$ chiral gauge theories and most attractive channels for bilinear fermion condensation. The quantities listed are k, N, and the respective ΔC_2 values for the $A_k \times A_k \to \bar{A}_{N-2k}$ and $A_k \times \bar{F} \to A_{k-1}$ channels. In the last column, we list the most attractive channel for bilinear fermion condensation in the strongly coupled and moderately strongly coupled (SC,MC) cases. If the UV to IR evolution remains weakly coupled (WC), it flows to a non-Abelian Coulomb phase (NACP). For $k = 2$, we include illustrative results including the interval $5 \le N \le 10$; for $k = 3,\ 4$ we list results for all (asymptotically free) $A_k\,\bar{F}$ theories. See text for further discussion of the $k = 2$, $N = 6$ and $k = 3$, $N = 12$ cases where the ΔC_2 values are equal. The $A_2\,\bar{F}$ theories could confine, yielding massless composite fermions. Possible multifermion condensates are also discussed in the text.

k	N	$\Delta C2(A_k \times A_k \to \bar{A}_{N-2k})$	$\Delta C_2(A_k \times \bar{F} \to A_{k-1})$	MAC for (S,M)C
2	5	4.800	3.600	$A_2 \times A_2 \to \bar{F}$
2	6	4.667	4.667	$A_2 \times \bar{F} \to F$
2	7	4.571	5.714	$A_2 \times \bar{F} \to F$
2	8	4.500	6.750	$A_2 \times \bar{F} \to F$
2	9	4.444	7.778	$A_2 \times \bar{F} \to F$
2	10	4.400	8.800	$A_2 \times \bar{F} \to F$
3	7	10.29	4.571	$A_3 \times A_3 \to \bar{F}$
3	8	10.125	5.625	$A_3 \times A_3 \to \bar{A}_2$
3	9	10.000	6.667	$A_3 \times A_3 \to \bar{A}_3$
3	10	9.900	7.700	$A_3 \times A_3 \to \bar{A}_4$
3	11	9.818	8.727	$A_3 \times A_3 \to \bar{A}_5$
3	12	9.750	9.750	$A_3 \times \bar{F} \to A_2$ or NACP
3	13	9.692	10.769	$A_3 \times \bar{F} \to A_2$ or NACP
3	14	9.643	11.786	$A_3 \times \bar{F} \to A_2$ or NACP
3	15	9.600	12.800	NACP
3	16	9.5625	13.8125	NACP
3	17	9.529	14.824	NACP
4	9	17.78	5.556	$A_4 \times A_4 \to \bar{F}$
4	10	17.60	6.600	$A_4 \times A_4 \to \bar{A}_2$ or NACP
4	11	17.45	7.636	NACP

6.5 $A_2\,\bar{F}$ Theories

6.5.1 General

In this section we analyze the UV to IR evolution of some $A_2\,\bar{F}$ theories in detail. Recall that the explicit fermion fields are $A_2: \ \psi_L^{ab}$ and $\bar{F}: \ \chi_{a,i,L}$, where $a, \ b$ are the SU(N) gauge indices and $i = 1, .., N-4$ is a copy (flavor) index. The one-loop and two-loop coefficients were given in Eqs. (6.7) and (6.20). We find that for all $N \ge N_{min} = 5$, the coefficient $b_2^{(2)}$ is positive, so the two-loop beta function of the $A_2\,\bar{F}$ theory has no IR zero. Hence, as the Euclidean reference scale μ decreases from the UV to the IR, the gauge coupling increases until it eventually exceeds the region where it is perturbatively calculable. This IR behavior is thus marked as SC, for strong coupling, in Table 6.1.

The global flavor symmetry group for this theory is given in Eq. (6.31) with the U(1)$'$ charge assignments in (6.32). This theory satisfies the 't Hooft global anomaly matching conditions [8,83], so, as it becomes strongly coupled in the infrared, it could confine and produce massless gauge-singlet composite spin-1/2 fermions as well as massive gauge-singlet mesons and also primarily gluonic states. If this happens, then it is a complete description of the UV to IR evolution. The three-fermion operator for the composite gauge-singlet fermion can be written as

$$f_{ij} \propto [\chi_{a,i,L}^T C \psi_L^{ab}]\chi_{b,j,L} + (i \leftrightarrow j) \ . \tag{6.57}$$

From Eq. (6.32), the U(1)$'$ charge of this composite fermion is

$$Q_{f_{ij}} = Q_{\bar{A}} + 2Q_{\bar{F}} = -N \ . \tag{6.58}$$

If $N \ge 6$, f_{ij} transforms as the conjugate symmetric rank-2 tensor representation, $\overline{\square\square}$, of the SU($N-4$)$_{\bar{F}}$ factor group in the global flavor symmetry group $G_{fl}^{(2)} = \text{SU}(N-4)_{\bar{F}} \otimes \text{U}(1)'$ of the theory.

Another possibility is that the SU(N) gauge interaction could produce bilinear fermion condensates, thereby breaking both gauge and global symmetries. The most attractive channel for this fermion condensation was determined, as a function of N, in Eq. (6.55). It can also be possible to form multifermion condensates involving more than two fermion fields, which preserve the chiral gauge symmetry. We will discuss this latter possibility in Sect. 6.8. Here we proceed to analyze bilinear fermion condensate formation for various specific theories.

6.5.2 SU(5) $A_2\,\bar{F}$ Theory

The simplest chiral gauge theory in the $A_2\,\bar{F}$ family of theories has the gauge group SU(5), with fermion content given by the $N = 5$ special case of Eq. (6.14), namely $A_2 + \bar{F} = [2]_5 + \overline{[1]}_5$. Like the other $A_2\,\bar{F}$ theories considered here that become strongly coupled in the infrared, this one could confine and produce a massless composite fermion. Alternatively, it could produce fermion condensates. The most attractive channel for bilinear fermion condensation in this theory is $A_2 \times A_2 \to \bar{A}_1$. If the dynamics is such that this condensate does, indeed, form, then we denote the mass scale at which it is produced as Λ_5. This condensate breaks the SU(5) gauge symmetry to SU(4). Without loss of generality, we take the gauge index corresponding to the breaking direction to be $a = 5$. The condensate then has the form

$$\langle \epsilon_{abde5}\psi_L^{ab\ T} C \psi_L^{de} \rangle \propto \Big[\langle \psi_L^{12\ T} C \psi_L^{34} \rangle - \langle \psi_L^{13\ T} C \psi_L^{24} \rangle + \langle \psi_L^{14\ T} C \psi_L^{23} \rangle \Big] \ . \tag{6.59}$$

The fermions involved in this condensate gain dynamical masses of order Λ_5, as do the nine gauge bosons in the coset SU(5)/SU(4). In addition to breaking the SU(5) gauge symmetry, the condensate has the nonzero value of the U(1)$'$ charge $Q'^{(2)} = -2$ given by the $k = 2$ special case of Eq. (6.50) and hence breaks the global U(1)$'$ symmetry. Since this symmetry is not gauged, this breaking yields one Nambu-Goldstone boson (NGB).

To construct the low-energy effective field theory with SU(4) chiral gauge invariance that describes the physics as the scale μ decreases below Λ_5, we decompose the fermion representations of SU(5) with respect to the unbroken SU(4) subgroup. It will be useful to give this decomposition more generally for SU(N) relative to an SU($N-1$) subgroup in our usual notation and also in terms of the corresponding Young tableaux:

$$[2]_N = \{[2]_{N-1} + [1]_{N-1}\} \ , \ i.e.,$$

$$\substack{\square\\ \square}_{\ \mathrm{SU}(N)} = \Big[\ \substack{\square\\ \square} + \square \ \Big]_{\ \mathrm{SU}(N-1)} \ . \tag{6.60}$$

The $[2]_4$ field is comprised of ψ_L^{ab} fermions with $1 \le a,\ b \le 4$ that gained dynamical masses of order Λ_5 and were integrated out of the low-energy theory. The other massless SU(4)-nonsinglet fermions are the $[1]_4 = F$ fermion ψ_L^{5b} with $1 \le b \le 4$ and the $\overline{[1]}_4 = \bar{F}$ fermion $\chi_{a,1,L}$ with $1 \le a \le 4$. Hence, the massless SU(4)-nonsinglet fermion content of this theory consists of $F + \bar{F}$, so this theory is vectorial. This SU(4) theory also contains the SU(4)-singlet fermion $\chi_{5,1,L}$. The one-loop and two-loop coefficients of the SU(4) beta function have the same sign, so again, this function has no IR zero, and therefore the SU(4) gauge coupling inherited from the SU(5) UV theory continues to increase as the reference scale μ decreases. Rewriting the left-handed $\bar{F}$ as a right-handed F, one sees that this is a vectorial SU(4) gauge theory with massless $N_f = 1$ Dirac fermion in the fundamental representation. It therefore has a classical global chiral flavor symmetry group $\mathrm{U}(1)_F \otimes \mathrm{U}(1)_{\bar{F}}$, or equivalently, $\mathrm{U}(1)_V \otimes \mathrm{U}(1)_A$ in standard notation. The $\mathrm{U}(1)_A$ is broken by SU(4) instantons, so the nonanomalous global flavor symmetry is $\mathrm{U}(1)_V$. At a scale $\Lambda_4 \lesssim \Lambda_5$, one expects that the SU(4) gauge interaction produces a bilinear fermion condensate in the most attractive channel, which is $F \times \bar{F} \to 1$, thus preserving the SU(4) gauge symmetry. The condensate is

$$\langle \sum_{b=1}^{4} \psi_L^{5b\ T} C \chi_{b,1,L} \rangle \ . \tag{6.61}$$

This condensate respects the $\mathrm{U}(1)_V$ global symmetry, and hence does not produce any Nambu-Goldstone bosons. Thus, this SU(4) theory confines and produces gauge-singlet hadrons (with the baryons being bosonic). In the infrared limit, the only remaining massless particles are the SU(4)-singlet fermion $\chi_{a,1,L}$ and the one Nambu-Goldstone boson resulting from the breaking of the $\mathrm{U}(1)'$ global flavor symmetry by the condensate (6.59).

6.5.3 SU(6) $A_2\,\bar{F}$ Theory

We next consider an SU(6) $A_2\,\bar{F}$ theory. The fermion content of this theory is the $N = 6$ special case of (6.14), namely $A_2 + 2\bar{F} = [2]_6 + 2\overline{[1]}_6$. The A_2 fermion is denoted $\psi_L^{ab} = -\psi_L^{ba}$, and the two copies of the $\bar{F}$ fermion are denoted $\chi_{a,i,L}$, where $1 \le a,\ b \le 6$ are gauge indices and $i = 1, 2$ is the copy index. We consider possible bilinear fermion condensates for this theory. As discussed above, although the bilinear fermion condensation channels $A_2 \times A_2 \to \bar{A}_2$ and $A_2 \times \bar{F} \to F$ have the same ΔC_2, a vacuum alignment argument favors the $A_2 \times \bar{F} \to F$ channel because it leaves a larger residual gauge symmetry, namely SU(5). Assuming that a condensate in this channel does form, we denote the scale at which it is produced as Λ_6. Again, by convention we take the breaking direction as $a = 6$ and the copy index as

$i = 2$ on the $\bar{F}$ fermion in the condensate, which can thus be written as

$$\langle \sum_{b=1}^{5} \psi_L^{6b\ T} C \chi_{b,2,L} \rangle \ . \tag{6.62}$$

This condensate also breaks the $\mathrm{SU}(2)_{\bar{F}} \otimes \mathrm{U}(1)'$ global flavor symmetry. The ψ_L^{6b} and $\chi_{b,2,L}$ fermions with $1 \le b \le 5$ involved in the condensate (6.62) get dynamical masses of order Λ_6, as do the 11 gauge bosons in the coset SU(6)/SU(5). These are integrated out of the low-energy effective SU(5)-invariant theory that describes the physics as the scale μ decreases below Λ_6.

From the $N = 6$ special case of the general decomposition (6.60) in conjunction with the form of the condensate (6.62), it follows that the massless SU(5)-nonsinglet fermion content of the descendant SU(5) theory is $A_2 + \bar{F}$, together with the (massless) SU(5)-singlet fermions $\chi_{6,1,L}$ and $\chi_{6,2,L}$. Thus, the SU(5)-nonsinglet fermion content of this theory is the same as that of the SU(5) theory discussed above, and our analysis there applies here. Since this SU(5) theory satisfies the 't Hooft global anomaly matching conditions, when it becomes strongly coupled, it could confine and produce massless SU(5)-singlet composite fermions, as well as massive mesons and primarily gluonic states, or it could self-break via fermion condensate formation. We also discuss below a possible SU(5)-preserving four-fermion condensate that might form.

6.5.4 SU(N) $A_2\,\bar{F}$ Theories with $N \ge 7$

For $N \ge 7$, the most attractive channel for bilinear fermion condensation is $A_2 \times \bar{F} \to F$, with ΔC_2 given by the $k = 2$ special case of (6.41),

$$\Delta C_2 = C_2([2]_N) = \frac{(N-2)(N+1)}{N} \quad \text{for } A_2 \times \bar{F} \to F \ . \tag{6.63}$$

The UV to IR evolution of these theories is similar to that of the SU(6) theory. At each stage, owing to the fact that the SU(N) theory and the various descendant theories satisfy 't Hooft global anomaly matching conditions, as the coupling gets strong in the IR, the gauge interaction may confine and produce massless composite fermions or may produce various fermion condensates. The most attractive channel for bilinear fermion condensation at a given stage is $A_2 \times \bar{F} \to F$, breaking the theory down to the next descendant low-energy theory. If the theory follows the first type of UV to IR flow, namely confinement with massless composite fermions, this extends all the way to the IR limit, while if the theory follows the second type of flow with condensate formation, then there is, in general, a resultant sequence of low-energy effective theories that describe the physics of the massless dynamical degrees of freedom at lower scales. If all of the stages involve gauge (and global) symmetry breaking by fermion condensates, then the gauge symmetry breaking is of the form

$$\mathrm{SU}(N) \ \to \ \mathrm{SU}(N-1) \to ... \to \mathrm{SU}(4) \ . \tag{6.64}$$

Here, the last theory, namely the SU(4) theory, is vectorial, while all of the higher-lying theories are chiral gauge theories.

6.6 $A_3 \bar{F}$ Theories

The fermion content of the $A_3 \bar{F}$ theories was displayed in Eq. (6.15). The one-loop and two-loop coefficients in the beta function were given in Eqs. (6.8) and (6.21), with numerical results for $\bar{b}_1$ and $\bar{b}_2$ displayed in Table 6.1. As is evident in Table 6.1, for $7 \le N \le 10$, the coefficient $\bar{b}_2$ is positive, so the two-loop beta function has no IR zero, and hence, as the reference scale μ decreases from large values in the UV toward the IR, the gauge coupling increases until it exceeds the region where it is perturbatively calculable. These theories are thus strongly coupled in the infrared (marked as SC in Table 6.1).

The next step in the analysis of the UV to IR flow in these theories is to determine if one or more of them might satisfy the 't Hooft global anomaly matching conditions. If this were to be the case, then, as in the $A_2 \bar{F}$ theories, one would have a two-fold possibility for the strongly coupled IR physics, namely confinement with gauge-singlet composite fermions but no spontaneous chiral symmetry breaking or formation of bilinear fermion condensates with associated breaking of gauge and global symmetries. For this purpose, we have examined possible SU(N) gauge-singlet fermionic operator products to determine if any of them could satisfy these global anomaly matching conditions. The global flavor symmetry group was given in Eq. (6.33) with (6.34). We have not found any such fermionic operator products. As an illustration of our analysis, let us consider the case $N = 7$, which contains a $[3]_7$ fermion ψ_L^{abd} and two fermions, $\chi_{a,i,L}$ with $i = 1, 2$ comprising two copies of the $\overline{[1]}_7$ representation. For this case,

$$G^{(3)}_{fl,N=7} = \mathrm{SU}(2)_{\bar{F}} \otimes \mathrm{U}(1)' \ , \tag{6.65}$$

with $\vec{Q}' = (1, -5)$. A fermionic operator product that is an SU(7) singlet is of the form

$$f_{i,R} = \epsilon_{abdefgh}[\psi_L^{abd\ T} C \psi_L^{efg}](\chi^c)^h_{i,R} \ , \tag{6.66}$$

where the c superscript denotes the charge conjugate fermion field. However, this vanishes identically. This can be seen as follows: an interchange (transposition) of ψ_L^{abd} and ψ_L^{efg} entails a minus sign from the switching of an odd number of indices in the antisymmetric SU(7) tensor density, a second minus sign from Fermi statistics, and a third minus sign from the fact that $C^T = -C$ for the Dirac charge conjugation matrix, so the operator is equal to minus itself and hence is zero.

Therefore, when theory becomes strongly coupled in the infrared, we will focus on the type of UV to IR evolution that leads to fermion condensates, and we consider bilinear fermion condensates here. The most attractive channel for these condensates, as a function of N, was given in Eq. (6.56).

As an explicit example of the $A_3 \bar{F}$ class of chiral gauge theories, let us consider the SU(7) theory, which has chiral fermion content given by the $N = 7$ special case of Eq. (6.15), namely

$$A_3 + 2\bar{F} = [3]_7 + 2\overline{[1]}_7 \ . \tag{6.67}$$

The most attractive channel for this theory is $A_3 \times A_3 \to \bar{F}$, which breaks the gauge symmetry SU(7) to SU(6) and also breaks the global flavor symmetry group $\mathrm{SU}(2)_{\bar{F}} \otimes \mathrm{U}(1)'$. We denote the scale at which this condensate forms as Λ_7. Without loss of generality, we label the gauge index for the broken direction to be $a = 7$. The condensate then has the form

$$\langle \epsilon_{abdefg7} \psi_L^{abd\ T} C \psi_L^{efg} \rangle \ . \tag{6.68}$$

Of the $\binom{7}{3} = 35$ components of the A_3 fermion, denoted generically as ψ_L^{abd}, the $\binom{7}{3} - \binom{6}{2} = 20$ components with $1 \le a,\ b,\ d \le 6$ that are involved in this condensate gain dynamical masses of order Λ_7, as do the 13 gauge bosons in the coset SU(7)/SU(6). These are integrated out of the low-energy effective theory SU(6) chiral gauge theory that describes the physics as the scale decreases below Λ_7.

The massless SU(6)-nonsinglet fermion content of this SU(6) theory thus consists of $A_2 + 2\bar{F} = [2]_6 + 2\overline{[1]}_6$, comprised by the $\binom{6}{2} = 15$ components ψ_L^{ab7} and the $\chi_{a,i,L}$ with $1 \le a,\ b \le 6$ and $i = 1, 2$. A theorem proved in [2] states that a low-energy effective theory that arises by dynamical symmetry breaking from an (asymptotically free) anomaly-free chiral gauge theory is also anomaly-free. One sees that the present example is in accord with this general theorem. Indeed, the nonsinglet fermions in this SU(6) descendant theory are precisely those of the SU(6) $A_2\,\bar{F}$ theory discussed above, and that analysis applies here for the further UV to IR evolution of the theory. In addition to the SU(6)-nonsinglet fermions, this descendant theory also contains the SU(6)-singlet fermions $\chi_{7,i,L}$ with $i = 1, 2$.

6.7 $A_4\,\bar{F}$ Theories

The fermion content of the $A_4\,\bar{F}$ theories was given in Eq. (6.16). The reduced one-loop and two-loop coefficients in the beta function were listed in Eqs. (6.9) and (6.22), with numerical results displayed in Table 6.1. We find that for each of the three relevant values of N, namely $N = 9,\ 10,\ 11$, the coefficient $\bar{b}_2$ is negative, so the two-loop beta function has an IR zero. As we noted above, for $N = 11$, this IR zero is at very weak coupling relative to the minimal critical value for bilinear fermion condensation, so we can reliably conclude that the theory evolves from the UV to a (deconfined) non-Abelian Coulomb phase in the infrared. In the $N = 9$ and $N = 10$ theories, the respective IR zeros in the two-loop beta function occur at strong and moderate coupling, so a full analysis is necessary.

We have examined whether there are SU(N) gauge-singlet composite fermion operators that could satisfy the 't Hooft global anomaly matching conditions, but we have not found any. The global flavor symmetry group was given in Eq. (6.35) with (6.36). As an illustration of our analysis, let us consider the SU(9) $A_4\,\bar{F}$ theory, which contains a $[3]_9$ fermion ψ_L^{abd} and the fermions, $\chi_{a,i,L}$ with $1 \le i \le 5$ comprising five copies of the $\overline{[1]}_9$ representation of SU(9). The global flavor symmetry group is

$$G^{(4)}_{fl,N=9} = {\rm SU}(5)_{\bar{F}} \otimes {\rm U}(1)' \ , \qquad (6.69)$$

The $\chi_{a,i,L}$ fermions transform as $\overline{\square}$ of the SU$(5)_{\bar{F}}$ flavor group, and the vector of U(1)$'$ charges is $\vec{Q}' = (Q'_{A_4}, Q'_{\bar{F}}) = (1, -7)$. A fermionic operator product that is an SU(9) gauge singlet is

$$f^i_R = \epsilon_{abdefghrs}[\psi^{abde\ T}_L C\psi^{fghr}_L](\chi^c)^{s,i}_R \ . \tag{6.70}$$

This transforms as a $\Box$ representation of the global $\mathrm{SU}(5)_{\bar F}$ symmetry with $\mathrm{U}(1)'$ charge $2Q'_{A_4} - Q'_{\bar F} = 9$. Since this is a right-handed composite fermion, we actually calculate with the charge conjugate $(f^c)_{i,L}$, which is a left-handed fermion that transforms as a $\overline{\Box}$ representation of the global SU(5) with $\mathrm{U}(1)'$ charge -9. We find that this composite fermion does not satisfy the global anomaly matching conditions. For example, consider the $\mathrm{SU}(5)^3$ anomaly. The fundamental fields make the following contributions: the A_4 fermion yields zero, while the $\bar F$ fermions yield $N\mathcal{A}(\overline{\Box}) = 9 \times (-1) = -9$. However, the f^c_L fermion yields $\mathcal{A}(\overline{\Box}) = -1$, which does not match. Since we have not found composite fermion operators that satisfy the 't Hooft global anomaly matching conditions, we consider fermion condensation in the cases where the beta function has an IR zero at moderate (for $N = 10$) and strong (for $N = 9$) coupling in the infrared.

As an explicit example, we analyze the SU(9) $A_4\,\bar F$ theory. The fermion content of this theory is given by the $N = 9$ special case of Eq. (6.16), namely

$$A_4 + 5\bar F = [4]_9 + 5\overline{[1]}_9 \ . \tag{6.71}$$

The most attractive channel for bilinear fermion condensation is the $N = 9$ special case of (6.48), namely $A_4 \times A_4 \to \bar F$. Assuming that this condensate forms, it breaks the gauge symmetry SU(9) to SU(8) and also breaks the global flavor symmetry group $\mathrm{SU}(5)_{\bar F} \otimes \mathrm{U}(1)'$. We denote the scale at which this condensate forms as Λ_9. Without loss of generality, we label the gauge index for the broken direction to be $a = 9$. The condensate then has the form

$$\langle \epsilon_{abdefghr9}\psi^{abde\ T}_L C\psi^{fghr}_L\rangle \ . \tag{6.72}$$

Of the $\binom{9}{4} = 126$ components of ψ^{abde}_L, the $\binom{9}{4} - \binom{8}{3} = 70$ components with $1 \le a,\ b,\ d,\ e \le 8$ that are involved in this condensate gain dynamical masses of order Λ_9, as do the 17 gauge bosons in the coset SU(9)/SU(8). These are integrated out of the low-energy effective theory SU(8) chiral gauge theory that describes the physics as the scale decreases below Λ_9.

The massless SU(8)-nonsinglet fermion content of this SU(8) theory thus consists of $A_3 + 5\bar F = [3]_8 + 5\overline{[1]}_8$, comprised by the $\binom{8}{3} = 56$ components ψ^{abd9}_L and the $\chi_{a,i,L}$ with $1 \le a,\ b,\ d \le 8$ and $1 \le i \le 5$. Again, the theorem proved in [2] guarantees that this SU(8) descendant theory is anomaly-free. Indeed, the nonsinglet fermions in this SU(8) descendant theory are precisely those of the SU(8) $A_3\bar{F}$ theory discussed above, and that analysis applies here for the further UV to IR evolution of the theory. In addition to the SU(8)-nonsinglet fermions, this descendant theory also contains the SU(6)-singlet fermions $\chi_{9,i,L}$ with $1 \le i \le 5$.

6.8 Multifermion Condensates and Implications for the Preservation of Chiral Gauge Symmetry

Our discussion above of fermion condensate formation focused on bilinear fermion condensates and resultant dynamical chiral gauge symmetry breaking. However, it is, in principle, possible for a strongly interacting vectorial or chiral gauge theory to produce fermion condensates involving product(s) of more than just two fermion fields [110, 111]. Much less attention as been devoted in the literature to such multifermion condensates than to bilinear fermion condensates. This is somewhat analogous to the situation with bound states of (anti)quarks in hadronic physics. For many years the main focus of research was on color-singlet bound states with the minimum number of (anti)quarks, namely qqq, for baryons and $q\bar{q}$ for mesons. (Subsequently, glueballs and mixing between $q\bar{q}$ mesons and glue to form mass eigenstates were also studied.) However, there is increasing experimental evidence that the hadron spectrum also contains bound states with additional quarks, such as $q\bar{q}q\bar{q}$ and $q\bar{q}Q\bar{Q}$, where Q means a heavy quark, c or b, including charged mesons, and possibly $qqqq\bar{q}$ and $qqqQ\bar{Q}$ [112]. In the case of possible condensates involving four or more fermions, we are not aware of a reliable method that can be used to assess the relative likelihood that these would form. The problem of assessing this likelihood is fraught with even more theoretical uncertainty than the uncertainty inherent in the use of the rough MAC criterion to measure the attractiveness of bilinear fermion condensation channels.

Clearly, Lorentz invariance implies that the number of fermion fields in such multifermion condensates must be even. As usual, we denote the charge conjugate of a generic fermion field χ as $\chi^c \equiv C\bar{\chi}^T$, where C is the Dirac charge conjugation matrix satisfying $C = -C^T$ and $\bar{\chi} \equiv \chi^\dagger\gamma_0$; recall also that for a left-handed fermion χ_L, the charge conjugate is $(\chi_L)^c = (\chi^c)_R$.

As an example, consider the SU(5) $A_2\,\bar{F}$ theory, with the fields ψ_L^{ab} and $\chi_{a,1,L}$ or equivalently, $\psi^c_{ab,R}$ and $(\chi^c)_R^{a,1}$. When the gauge interaction becomes strong, it could produce several different four-fermion condensates that preserve the SU(5) gauge symmetry. One such condensate that involves all of the fermions is

$$\langle \epsilon_{abdef}[\psi_L^{ab\ T} C \psi_L^{de}][\psi_L^{fs\ T} C \chi_{s,1,L}]\rangle \ , \tag{6.73}$$

where a,b,d,e,f,s are SU(5) gauge indices. This condensate has U(1)′ charge $3Q'_{A_2}+Q'_{\bar{F}}$. Using the results from the $N=5$ special case of Eq. (6.32), namely, $Q'_{A_2}=1$, $Q'_{\bar{F}}=-3$, we find that this condensate (6.73) has zero U(1)′ charge, so it also preserves the global U(1)′ symmetry of the SU(5) theory.

In a similar manner, consider the SU(6) $A_2\,\bar{F}$ theory, with the fermions ψ_L^{ab} and $\chi_{a,j,L}$ with $j=1,2$. As the SU(6) gauge interaction becomes strong in the infrared, it might produce the following four-fermion condensate that is invariant under the SU(6) gauge symmetry:

$$\begin{aligned} & \langle \epsilon_{abdesu}[\psi_L^{ab\ T} C \psi_L^{de}][(\chi^c)_R^{s,1\ T} C (\chi^c)_R^{u,2}]\rangle - (1 \ \leftrightarrow \ 2) \\ = \quad & \langle \epsilon_{abdesu}[\psi_L^{ab\ T} C \psi_L^{de}]\epsilon_{ij}[(\chi^c)_R^{s,i\ T} C (\chi^c)_R^{u,j}]\rangle \ . \end{aligned} \tag{6.74}$$

Note that because of the contraction of the operator product $[(\chi^c)_R^{s,1\ T} C (\chi^c)_R^{u,2}]$ with the SU(6) ϵ_{abdesu} tensor, the first term in Eq. (6.74) is automatically antisymmetrized in the flavor indices $j=1,2$; we have made this explicit by subtracting the term with these indices interchanged. As shown by the second line of Eq. (6.74), this condensate thus preserves the $SU(2)_{\bar{F}}$ factor group in the global flavor symmetry G_{fl} for this theory, namely $SU(2)_{\bar{F}} \otimes U(1)'$. In the $(A_2,\bar{F})$ basis, the U(1)′ charges are $(2,-4)$, as given by the $N=6$ special case of Eq. (6.32). Hence, the U(1)′ charge of the condensate (6.74) is -4, so it breaks the U(1)′ part of G_{fl}, yielding one Nambu-Goldstone boson.

One can give corresponding discussions of gauge-invariant multifermion condensates for other SU(N) $A_k\,\bar{F}$ theories that become strongly coupled in the infrared. In general, these theories could also produce other types of four-fermion condensates such as

$$\langle [\psi_L^{ab\ T} C \chi_{b,i,L}][(\psi^c)^T_{ad,R} C (\chi^c)_R^{d,j}]\rangle \ , \tag{6.75}$$

$$\langle [\bar{\psi}_{ab,L}\gamma_\mu \psi_L^{ab}][\bar{\psi}_{de,L}\gamma^\mu \psi_L^{de}]\rangle \ , \tag{6.76}$$

$$\langle [\bar{\psi}_{ab,L}\gamma_\mu \psi_L^{ab}][(\bar{\chi}_L)^{d,i}\gamma^\mu \chi_{d,j,L}]\rangle \ , \tag{6.77}$$

and

$$\langle [(\bar{\chi}_L)^{a,i}\gamma_\mu \chi_{a,j,L}][(\bar{\chi}_L)^{b,k}\gamma^\mu \chi_{b,\ell,L}]\rangle \ , \tag{6.78}$$

where $1 \le i, \ j, \ k, \ \ell \le n_{\bar{F}}$. There are also multifermion condensates with eight and more fermions that one could consider. Such multifermion condensates merit further study.

6.9 Non-Existence of Asymptotically Free $S_k \, \bar{F}$ Theories with $k \ge 3$

It is natural to carry out an investigation of (anomaly-free) chiral gauge theories with gauge group SU(N) and chiral fermions transforming according to the rank-k symmetric tensor representation with $k \ge 3$ and a requisite number of chiral fermions in the $\bar{F}$ representation so as to render the theories free of an anomaly in gauged currents. We denote such a theory as an SU(N) $S_k \, \bar{F}$ theory. This investigation would be the analogue of the study that we have performed in this chapter for $A_k \, \bar{F}$ theories with $k \ge 3$ and would generalize the studies that have been carried out in the past on the $S_2 \, \bar{F}$ theory [1, 11, 22, 82, 83]. As with the $A_k \, \bar{F}$ theories, we require that the theory must be asymptotically free so that it is perturbatively calculable in at least one regime, namely the deep UV, where the gauge coupling is small.

However, we shall show here that there are no asymptotically free (anomaly-free) $S_k \, \bar{F}$ chiral gauge theories with $k \ge 3$. As before we denote the number of copies of $\bar{F}$ fermions as $n_{\bar{F}}$. The contribution to the triangle anomaly in gauged currents of a chiral fermion in the S_k representation is (see Appendix A)

$$\mathcal{A}(S_k) = \frac{(N+k)! \, (N+2k)}{(N+2)! \, (k-1)!} \ . \tag{6.79}$$

The total anomaly in the theory is $\mathcal{A} = \mathcal{A}(S_k) - n_{\bar{F}}\mathcal{A}(F)$, so the condition of anomaly cancellation is that

$$n_{\bar{F}} = \mathcal{A}(S_k) \ . \tag{6.80}$$

The first few values are of $n_{\bar{F}}$ are

$$n_{\bar{F}} = \begin{cases} N+4 & if\text{k=2} \\ (1/2)(N+3)(N+6) & \text{if } k=3 \\ (1/6)(N+3)(N+4)(N+8) & \text{if } k=4 \end{cases} \tag{6.81}$$

and so forth for higher k.

To investigate the restrictions due to the requirement of asymptotic freedom, we calculate the one-loop coefficient of the beta function. We find

$$\begin{aligned} b_{1,S_k\bar{F}} &= \frac{1}{3}\Big[11N - 2\Big\{T(S_k) + \mathcal{A}(S_k)T(\bar{F})\Big\}\Big] \\ &= \frac{1}{3}\Big[11N - \frac{1}{(k-1)!}\Big\{\Big[\prod_{j=2}^{k}(N+j)\Big] + \frac{(N+k)!(N+2k)}{(N+2)!}\Big\}\Big] . \end{aligned} \tag{6.82}$$

We exhibit the explicit expressions for $b^{(k)}$ for the first few $k \geq 2$:

$$b_{1,S_2\bar{F}} = 3N + 2 , \tag{6.83}$$

$$b_{1,S_3\bar{F}} = \frac{1}{3}(-N^2 + 4N - 12) , \tag{6.84}$$

$$b_{1,S_4\bar{F}} = -\frac{1}{9}(N^3 + 12N^2 + 14N + 60) , \tag{6.85}$$

$$b_{1,S_5\bar{F}} = -\frac{1}{36}(N^4 + 18N^3 + 119N^2 + 210N + 360) , \tag{6.86}$$

and

$$b_{1,S_6\bar{F}} = -\frac{1}{180}\Big(N^5 + 25N^4 + 245N^3 + 1175N^2 + 2094N + 2520\Big) . \tag{6.87}$$

The coefficient $b_{1,S_2\bar{F}}$ is positive for all relevant N, and this property was used in past studies of the $S_2\,\bar{F}$ theory. However, the coefficient $b_{1,S_3\bar{F}}$ is negative for relevant $N \geq 3$. (Recall that an SU(2) theory has only real representations and hence is not chiral.) With N generalized from positive integers to real numbers, $b_{1,S_3\bar{F}}$ is negative for all N, reaching its maximum value of $-8/3$ for $N = 2$. We find that the $b_{1,S_k\bar{F}}$ coefficients with $k \geq 4$ are negative-definite for all positive N (either real or integer). This is evident from the illustrative explicit expressions that we have given for $4 \leq k \leq 6$. This completes our proof that there are no asymptotically free, (anomaly-free) $S_k\,\bar{F}$ chiral gauge theories with $k \geq 3$.

Chapter 7

Dynamical Symmetry Breaking in Chiral Gauge Theories with Direct-Product Gauge Groups

In this chapter, we summarize the analysis in [4], in which we studied patterns of dynamical gauge symmetry breaking in strongly coupled chiral gauge theories with direct-product gauge groups of the form

$$G = \bigotimes_{j=1}^{N_G} G_j \ . \tag{7.1}$$

From comparative studies of a number of different theories of this type, we infer some common features, including the property that the symmetry-breaking behavior depends sensitively on the relative sizes of the gauge couplings of the different factor groups in the direct product.

7.1 Classification of Groups and Methods of Analysis

In order to explore the nonperturbative behavior of direct-product chiral gauge theories, it is useful to have a general classification of these theories and general methods for analyzing them. We discuss these in this section. As stated above, we consider direct-product chiral gauge theories with gauge groups of the form (7.1) with fermion content $\{f\}$ chosen such that the theory is free of any anomalies in gauged currents and free of any global SU(2) Witten anomalies, and also such that all non-Abelian gauge interactions are asymptotically free. The reason for the latter property is that this ensures that at sufficient high Euclidean scale μ in the UV, one has perturbative control over the theory. Unless otherwise noted, we will, with no loss of generality, write all fermions as left-handed chiral components.

To establish our classification system, we first introduce some notation. We generically denote a group that has only real or pseudoreal representations as G_r and a group that has complex representations as G_c. A group G_r cannot, by self, be the gauge group of a chiral gauge theory, although it can appear as a factor group in a chiral gauge theory. A group G_r has zero anomaly, while, in general, a group G_c has nonzero anomalies $A_{\mathcal{R}}$ for its representations (see Eq. (A.20)), which we will indicate by the symbol G_{ca}. If a group G_c has no anomaly, i.e., $A_{\mathcal{R}} = 0$ for all $\mathcal{R}$, then it is commonly termed "safe" (s) [106, 114], and we denote it as G_{cs}. Of course, a group G_r is automatically safe. Thus, the generic class G_s includes G_r and G_{cs}.

We may then classify a chiral gauge theory with the direct-product gauge group (7.1) by an N_G-dimensional vector indicating the nature of the factor groups involved in the direct product. If $N_G = 1$, there are two possibilities: (i) (ca), e.g., SU(N) with $N \geq 3$, and (ii) (cs), e.g., SO($4k + 2$) for $k \geq 2$ or the exceptional group E_6 [104, 106, 114, 115]. For $N_G = 2$, the possibilities are

$$N_G = 2 : \quad (ca, r),\ (cs, r),\ (ca, ca),\ (ca, cs),\ (cs, cs)\ , \tag{7.2}$$

where we do not distinguish the order of factor groups, so (cs, ca) and (ca, cs) are the same type, etc.

Let us consider a factor group G_i in (7.1) which is of the form G_{ca}, and set the gauge couplings of the other factor groups to zero. If the resultant G_{ca} theory is vectorial (v), then we denote this as G_{cav}. This is the case, for example, with the color SU(3) factor group in the Standard Model. Thus, a further classification of direct-product chiral gauge theories can be carried out in which, for for each factor group of the form G_{ca}, one distinguishes whether or not it is of the form G_{cav}. The Standard Model gauge group is of the type (cav, r, ca) in this classification. We summarize the classification of some chiral gauge theories considered in this chapter in Table 7.1.

Our requirement that each non-Abelian factor group in the direct product (7.1) is asymptotically free enables us to describe the theory perturbatively in the deep ultraviolet. We discuss the evolution from the UV to the IR next.

Table 7.1: Classification of some direct-product chiral gauge theories. See text for further discussion.

type	N_G	G
(ca,r), (cav,r)	2	$\mathrm{SU}(N) \otimes \mathrm{SU}(2)$ with $N \geq 3$
(cav,cav)	2	$\mathrm{SU}(N) \otimes \mathrm{SU}(M)$ with $N, M \geq 3$
(r,ca)	2	$\mathrm{SU}(2) \otimes \mathrm{U}(1)$
(ca,ca), (cav,ca)	2	$\mathrm{SU}(N) \otimes \mathrm{U}(1)$ with $N \geq 3$
(cav,r,ca)	3	$\mathrm{SU}(N_c) \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$
(cav,r,r)	3	$\mathrm{SU}(N) \otimes \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$
(cav,r,r,cav)	4	$\mathrm{SU}(N_c) \otimes \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_{B-L}$
(cs,r)	2	$\mathrm{SO}(4k+2) \otimes \mathrm{SU}(2)$ with $k \geq 2$
(cs,cav)	2	$\mathrm{SO}(4k+2) \otimes \mathrm{SU}(N)$ with $N \geq 3$
(cs,cs)	2	$\mathrm{SO}(4k+2) \otimes \mathrm{SO}(4k'+2)$ with $k, k' \geq 2$

To each factor group G_i, $i = 1, ..., N_G$, there corresponds a running gauge coupling $g_i(\mu)$, and we define $\alpha_i(\mu) = g_i(\mu)^2/(4\pi)$ and $a_i(\mu) \equiv g_i(\mu)^2/(16\pi^2)$. The argument μ will often be suppressed in the notation. The UV to IR evolution of the gauge coupling is determined by the beta function, $\beta_{g_i} = dg_i/dt$, or equivalently, $\beta_{G_i} = d\alpha_i/dt = [g/(2\pi)]\beta_{g_i}$, where $dt = d\ln\mu$. This has the series expansion

$$\beta_{G_i} = -8\pi a_i^2 \left[b_{G_i,1\ell} + \sum_{j=1}^{N_G} b_{G_i,2\ell;ij} a_j + \sum_{j,k=1}^{N_G} b_{G_i,3\ell;ijk} a_j a_k + ... \right] , \qquad (7.3)$$

where an overall minus sign is extracted and the ... indicate higher-loop terms. Here, $b_{G_i,1\ell}$ is the one-loop (denoted (1ℓ)) coefficient, multiplying a_i^2, $b_{G_i,2\ell;ij}$ is the two-loop coefficient, multiplying $a_i^2 a_j$, and so forth for higher-loop loops. The property of asymptotic freedom for the non-Abelian gauge interactions means that $\beta_{G_i} < 0$ for small α_i, $i = 1, ..., N_{NA}$. The set (7.3) constitutes a set of N_G coupled nonlinear first-order ordinary differential equations for the quantities α_i, $i = 1, ..., N_G$. To leading order, i.e., to one-loop order, the set of differential equations decouple from each, and one has the simple solution for each $i \in \{1, ..., N_G\}$:

$$\alpha_i(\mu_1)^{-1} = \alpha_i(\mu_2)^{-1} - \frac{b_{G_i,1\ell}}{2\pi} \ln\left(\frac{\mu_2}{\mu_1}\right) , \qquad (7.4)$$

where we take $\mu_1 < \mu_2$.

In the following discussion, we assume that the fundamental Lagrangian has no fermion mass terms, so that all fermion masses are generated dynamically by chiral symmetry breaking. For a pair of gauge interactions corresponding to the factor groups G_i and G_j in Eq. (7.1), the respective beta functions β_{G_i} and β_{G_j} in the deep UV are fixed once we choose the fermion content of a given theory. The values of the corresponding $\alpha_i(\mu_1)$ and $\alpha_j(\mu_1)$ at lower Euclidean scales are determined by (i) the initial values of $\alpha_i(\mu_2)$ and $\alpha_j(\mu_2)$ in the UV; (ii) the values of β_{G_i} and β_{G_j}; and (iii) the occurrence of bilinear fermion condensate formation at some scale(s) as the theory evolves from the deep UV toward the IR, which produce dynamical masses for the fermions involved in these condensates. Since we do not assume that the direct-product group (7.1) is contained in a simple group in the deep UV, we are free to consider various different orderings of the sizes of the couplings $\alpha_i(\mu_2)$ in the UV. Furthermore, because of the condensation process(es) (i-ii), the fermions involved in these condensates, together with gauge bosons corresponding to broken generators of gauge symmetries, acquire dynamical masses and are integrated out of the low-energy effective field theories that are applicable as the Euclidean reference scale decreaes below each condensation scale. The reduction in massless particle content in (iii) produces changes in the beta functions of the gauge interactions involved. Because of this, even if $\beta_{G_i} > \beta_{G_j}$ with all fermions initially present in the deep UV, it can happen that at a lower scale this inequality is reversed. The variation of gauge couplings in the deep UV embodied in the input (i) above was carried out in the earlier work [20] where both of the cases of relative sizes of α_{ETC} and α_{HC} were considered, and in [113], where both of the cases of relative sizes of couplings for $SU(3)_c$ and $SU(2)_L$ were considered. Henceforth, for notational simplicity, we set $b_{G_i,1\ell} \equiv b_{G_i,1}$. Here, as in the earlier works involving gauge theories with multiple gauge couplings [20, 25, 113], we will focus on the nonperturbative phenomenon of fermion condensate formation and the associated pattern of gauge symmetry breaking. The one-loop result (7.4) will be sufficient for our purposes here since we focus on this nonperturbative fermion condensate formation. These condensates also generically break global chiral symmetries, and we will give some examples of this.

In general, a fermion condensate may involve different fermion fields or the same fermion field. If the fields are the same, we may write the bilinear fermion operator product abstractly as follows. Assume that the gauge group G in Eq. (7.1) contains $t \le N_G$ non-Abelian factors G_k and that the relevant fermion field f transforms as the representation $\mathcal{R} \equiv (\mathcal{R}_1, ..., \mathcal{R}_t)$ under the direct product of these non-Abelian factor groups. Then the bilinear fermion product of a given fermion field is

$$f^T_{\mathcal{R},i,L} C f_{\mathcal{R},j,L} \ , \tag{7.5}$$

where C is the Dirac conjugation matrix, gauge group indices are suppressed in the noation, and i, j are copy (flavor) indices. From the property $C^T = -C$ together with the anticommutativity of fermion fields, it follows that the bilinear fermion operator product (7.5) is symmetric under interchange of the order of fermion fields and therefore is symmetric in the overall product

$$\Big[\prod_{k=1}^{t} (R_k \times R_k) \Big] \, S_{ij} \ , \tag{7.6}$$

where S_{ij} abstractly denotes the symmetry property under interchange of flavors, with $S_{ij} = (ij)$ and $S_{ij} = [ij]$ for symmetric and antisymmetric flavor structure, respectively. For example, for the case $t = N_g = 2$ and flavor indices i, j, the symmetry property (7.6) means that $f^T_{i,L} C f_{j,L}$ is of the form (s, s, s), (s, a, a), (a, s, a), or (a, a, s), where s and a indicate symmetric and antisymmetric and the three entries refer to the representations R_1 of G_1, R_2 of G_2, and S_{ij}. Thus, as an illustration, in the last case, (a, a, s), the product (7.5) would transform as antisymmetric representations in the Clebsch-Gordan products of $R_j \times R_j$ for $j = 1, 2$ and would be symmetric in flavor indices, with $S_{ij} = (ij)$, and so forth for other cases.

The main perturbative information that we will use is the one-loop coefficients of the beta functions for the non-Abelian gauge interactions. We require that these interactions must be asymptotically free so that we have perturbative control over them in the deep UV. If $\alpha_i(\mu)$ becomes strong, i.e., O(1) in the IR, one can no longer use perturbative methods reliably, but one can make use of several approximate methods to explore possible nonperturbative properties of the theory. First, one may investigate whether the fermions in the theory satisfy the 't Hooft anomaly-matching conditions. For this purpose, one determines the global flavor symmetry group of the theory is invariant and then checks whether candidate operators for gauge-singlet composite spin-1/2 fermions match the anomalies in the global flavor symmetries. If this necessary condition is satisfied, then it is possible that in the infrared the strong chiral gauge interaction could confine and produce massless composite spin-1/2 fermions.

A different possibility in a strongly coupled chiral gauge theory is that the gauge interaction can produce bilinear fermion condensates. This will be the main focus of our analysis here. In an irreducibly chiral theory these condensates break one or more gauge symmetries, as well as global flavor symmetries. A commonly used method for suggesting which type of condensate is most likely to form in this case is the most-attractive-channel (MAC) method [18]. For possible condensation of chiral fermions in the representations $R_{G_i,1}$ and $R_{G_i,2}$ of the factor group G_i in (7.1) in various channels of the form $R_{G_i,1} \times R_{G_i,2} \to R_{G_i,cond.}$, the MAC approach predicts that the condensation will occur in the channel with the largest (positive) value of the quantity $\Delta C_2 \equiv C_2(R_{G_i,1}) + C_2(R_{G_i,2}) - C_2(R_{G_i,cond.})$. This is only a rough measure, based on one-gluon exchange. The form of the condensate determines the resultant symmetry and form of vacuum alignment [102].

7.2 Methods for Constructing Chiral Gauge Theories

In this section we mention some useful methods for constructing anomaly-free direct-product chiral gauge theories.

7.2.1 Reduction Method

Let us say that we have a chiral gauge theory with the N_G-fold direct product gauge group (7.1) and a given fermion content that satisfies the constraints that the theory must be free of any anomaly in gauged currents, any possible global SU(2) anomaly, and, if G includes abelian factor groups, also any mixed gravitational-gauge anomaly. One can then construct a set of chiral gauge theories by a process of reduction, setting one or more of the gauge couplings $\{g_1, ..., g_{N_G}\}$ equal to zero. (ii) by turning off the $\mathrm{SU}(2)_L$ gauge coupling, one gets an $\mathrm{SU}(N_c) \otimes \mathrm{U}(1)_Y$ gauge theory of type (cav, ca); and (iii) by turning off the $\mathrm{U}(1)_Y$ coupling, one gets an $\mathrm{SU}(N_c) \otimes \mathrm{SU}(2)_L$ gauge theory of type (cav, r). Given that the original theory has the requisite property that all non-Abelian gauge interactions are asymptotically free, the theory derived by turning off some gauge coupling(s) also has this property.

7.2.2 Extension Method to Construct $G = \tilde{G} \otimes G_s$ Theories

Here we present a method for constructing a direct-product chiral gauge theory with an $(N_G + 1)$-fold direct-product gauge group, starting from a given chiral gauge theory with an N_G-fold direct-product gauge group $\tilde{G}$ by adjoining a safe group G_s to $\tilde{G}$ to produce

$$G = \tilde{G} \otimes G_s \tag{7.7}$$

and extending the fermion representations of $\tilde{G}$ to those of $G = \tilde{G} \otimes G_s$. Here G_s may be G_r or G_{cs}. The procedure is as follows:

1. Start with an anomaly-free chiral gauge theory with the N_G-fold gauge group $\tilde{G} = \bigotimes_{i=1}^{N_G} G_i$ and a set of fermion representations $\{\mathcal{R}_{\tilde{G}}\}$, where each of these is
$$\mathcal{R}_{\tilde{G}} = (\mathcal{R}_{G_1}, ..., \mathcal{R}_{G_{N_G}}) \ . \tag{7.8}$$

2. Choose the safe group G_s, of type G_r or G_{cs}, i.e., either a group with real representations, such as SU(2), or a safe group with complex representations, such as SO$(4k+2)$ with $k \geq 2$ or the exceptional group E_6.

3. Extend each fermion representation $\mathcal{R}_{\tilde{G}}$ of $\tilde{G}$ to a representation $\mathcal{R}_G$ of G using a single representation $\mathcal{R}_{G_s}$ of G_s to form $\mathcal{R}_G = (\mathcal{R}_{\tilde{G}}, \mathcal{R}_{G_s})$. As far as the $\tilde{G}$ group is concerned, this simply amounts to a replication of its original (anomaly-free) fermion content by $\dim(\mathcal{R}_{G_s})$ copies, so the resulting extended fermion content is also anomaly-free.

4. Apply the constraint that if the safe group is $G_s = \text{SU}(2)$, then the resultant theory must be free of a global SU(2) Witten anomaly associated with the homotopy group $\pi_4(\text{SU}(2)) = \mathbb{Z}_2$ [116, 117]. With $R_{G_s} = \square$, the necessary and sufficient condition to satisfy this constraint is that the total number of SU(2) doublets is even [116, 118].

5. Apply the constraints that each of the gauge interactions corresponding to non-Abelian factor groups in $\tilde{G}$ must remain asymptotically free in the larger group G, and the G_s gauge interaction must also be asymptotically free.

This method can be used to construct many types of direct-product chiral gauge groups. Among the $N_G = 2$ cases, for example, these types include all of the ones listed in Eq. (7.2).

7.3 $G_{cav} \otimes \mathrm{SU}(2)$ Theories

In this section we construct and study a class of $N_G = 2$ direct-product chiral gauge theories with a gauge group

$$G_1 \otimes G_2 = G_{cav} \otimes \mathrm{SU}(2) \ . \tag{7.9}$$

This class is the special case (cav, r) of the class $G_{ca} \otimes G_r$ discussed in Section 7.1 in which $G_{ca} = G_{cav}$, i.e., G_{ca} is a group with complex representations and $\mathcal{A}_{\mathcal{R}} \neq 0$ and the fermion content is such that if the SU(2) gauge interaction is turned off, then the G_{cav} gauge interaction is vectorial. This property guarantees that there is no cubic triangle anomaly in gauged currents in the G_{cav} sector. Furthermore, as already indicated above, since SU(2) has (pseudo)real representations, it has no anomaly. The only anomaly constraint is then the requirement that the SU(2) group must be free of a global anomaly. We consider theories of this type with chiral fermion content (written here as left-handed)

$$\{f_{ns,ns}\} = \sum_{\mathcal{R}} p_{\mathcal{R}} \, (\mathcal{R}, \square) \ , \tag{7.10}$$

$$\{f_{ns,s}\} = 2 \sum_{\mathcal{R}} p_{\bar{\mathcal{R}}} \, (\bar{\mathcal{R}}, 1) \ , \tag{7.11}$$

and optionally,

$$\{f_{s,ns}\} = p_1 \, (1, \square) \ , \tag{7.12}$$

where the subscripts ns and s are abbreviations for "nonsinglet" and "singlet"; $\mathcal{R}$ denotes a (nonsinglet) representation of the group G_1; and the first and second entries in subscripts and in the parentheses refer to the representations of G_{cav} and $\mathrm{SU}(2)_L$, respectively, with $\square$ being the fundamental representation in standard Young tableaux notation.

If the fermion sector includes only a single $\mathcal{R}$, then we set $p_{\mathcal{R}} \equiv p$ for brevity. We shall use interchangeably a notation with Young tableaux and dimensionalities to identify the representation: $(\mathcal{R}, \square) \leftrightarrow (\dim(\mathcal{R}), 2)$. In general, we will allow for several types of (nonsinglet) representations $\mathcal{R}$, but will focus on minimal theories with only one $\mathcal{R}$. The subscript indices i, j are copy ("flavor") indices, and the total number of copies of the $f_{ns,ns}$ fermions transforming as the $\mathcal{R}$ representation of G_1 is denoted $p_{\mathcal{R}}$. We shall mainly focus on irreducibly chiral theories, i.e., those for which the chiral gauge theory forbids any bare mass terms, but we shall also discuss some chiral gauge theories with vectorlike subsectors. The global symmetries depend on p and p_1; we will discuss them for specific models below.

The number of SU(2) chiral fermion doublets in this theory, which we shall denote N_d, is

$$N_d = p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R}) \ . \tag{7.13}$$

The condition that the SU(2) gauge sector must be free of a global anomaly is that

$$N_d \ \text{ is } \ \text{even} \ . \tag{7.14}$$

Because N_d is necessarily even, one could take half of the left-handed SU(2)-doublet fermions, rewrite them as right-handed charge-conjugates, and thereby put the SU(2) gauge interaction into vectorial form.

As noted, we shall also impose two further requirements on the theory, namely that the G_1 and the SU(2) gauge interactions must both be asymptotically free. From the general results in [75], we find that the one-loop coefficient of the beta function of the G_1 gauge interaction is

$$b_{1,G_1} = \frac{1}{3}\Big[11C_2(G_1) - 8\sum_{\mathcal{R}} p_{\mathcal{R}}\, T(\mathcal{R})\Big] \ , \tag{7.15}$$

so the requirement that the G_1 gauge interaction should be asymptotically free implies that

$$\sum_{\mathcal{R}} p_{\mathcal{R}}\, T(\mathcal{R}) < \frac{11C_2(G_1)}{8} \ . \tag{7.16}$$

Here and below, if $p_1 = 0$ and the theory contains fermions in one (nonsinglet) representation $\mathcal{R}$ of G_1, then only nonzero values of $p_{\mathcal{R}} \equiv p$ are relevant, since if $p = 0$, then the theory is a pure (direct-product) gauge theory and hence is not a chiral gauge theory.

The one-loop coefficient of the beta function of the SU(2) gauge interaction is

$$b_{1,\mathrm{SU}(2)_L} = \frac{1}{3}(22 - N_d)$$

$$= \frac{1}{3}\Big[22 - \Big(p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R})\Big)\Big] \ , \tag{7.17}$$

so the requirement that the SU(2) gauge interaction should be asymptotically free implies that

$$p_1 + \sum_{\mathcal{R}} p_{\mathcal{R}} \dim(\mathcal{R}) < 22 \ . \tag{7.18}$$

7.4 $\mathrm{SU}(N) \otimes \mathrm{SU}(2)$ Theories

In this section we construct and study several models with a direct-product gauge group of the form (7.9) with the first gauge group being $\mathrm{SU}(N)$, i.e., with

$$G = G_1 \otimes G_2 = \mathrm{SU}(N) \otimes \mathrm{SU}(2) \tag{7.19}$$

and various chiral fermion contents, which we denote as Models A, B, and C. All three of these models are of type (cav, r), as indicated in Table 7.1.

7.4.1 Model A

The first model that we consider, denoted Model A, is a minimal one in three respects: (i) it contains no G_1-singlet fermions, i.e., $p_1 = 0$; (ii) the fermions transform according to only one representation $\mathcal{R}$ of G_1 and its conjugate; and (iii) this representation $\mathcal{R}$ is the simplest nontrivial one, namely the fundamental, $\mathcal{R} = \Box$. The chiral fermions are

$$\psi^{a,\alpha}_{i,L}, \ i = 1, ..., p \ : \quad p\,(\Box, \Box) = p\,(N, 2), \tag{7.20}$$

and

$$\chi_{a,j,L}, \ j = 1, ..., 2p \ : \quad 2p\,(\bar{\Box}, 1) = 2p\,(\bar{N}, 1) \ . \tag{7.21}$$

Here, a and α are $\mathrm{SU}(N)$ and SU(2) gauge indices and i, j are copy ("flavor") indices. For $N \geq 3$, the chiral gauge symmetry forbids any bare mass terms for the fermions. In contrast, if $N = 2$, then gauge-invariant bare mass terms such as

$$\epsilon^{ab} \chi^T_{a,i,L} C \chi_{b,j,L} \ , \quad i \neq j, \quad 1 \leq i, j \leq 2p \tag{7.22}$$

and

$$\epsilon_{ab} \epsilon_{\alpha\beta} \psi^{a,\alpha \ T}_{i,L} C \psi^{b,\beta}_{j,L} \ , \quad 1 \leq i, j \leq p \tag{7.23}$$

can occur. Closely related to this, if $N = 2$, then the SU(N) and SU(2) gauge interactions can both be written in vectorial form, so the theory is not a chiral gauge theory. Therefore, henceforth we shall assume that $N \geq 3$ for this class of theories. In the notation introduced above, the fermion content of this Model A can be categorized as being of the form

$$\{f_{ns,ns}, \ f_{ns,s}\} \ . \tag{7.24}$$

The fermion terms in the Lagrangian for this model are

$$\mathcal{L} = \sum_{i=1}^{p} \bar{\psi}_{a,\alpha,i,L} i \not{D} \psi^{a,\alpha}_{i,L} + \sum_{j=1}^{2p} \bar{\chi}^{a}_{j,L} i \not{D} \chi_{a,j,L} \ , \tag{7.25}$$

(where we have indicated the sums over flavor explicitly).

For this Model A, the condition that the SU(2) gauge sector should be free of a global anomaly is

$$N_d = pN \text{ is even,} \tag{7.26}$$

and we require that this condition must be satisfied.

From the general result (7.15), we have, for the one-loop coefficient of the SU(N) beta function,

$$b_{1,\mathrm{SU}(N)} = \frac{1}{3}(11N - 4p) \ . \tag{7.27}$$

Therefore, the requirement that the SU(N) gauge interaction should be asymptotically free, expressed by the inequality (7.16), reads

$$p < \frac{11N}{4} \ . \tag{7.28}$$

From the general result (7.17), we find, for the one-loop coefficient of the SU(2) beta function,

$$b_{1,\mathrm{SU}(2)_L} = \frac{1}{3}(22 - pN) \ . \tag{7.29}$$

Hence, the requirement that the SU(2) gauge interaction should be asymptotically free, given by the inequality (7.18), is

$$p\,N < 22 \ . \tag{7.30}$$

Table 7.2: Values of N and p in the Model A $\mathrm{SU}(N) \otimes \mathrm{SU}(2)_L$ chiral gauge theory allowed by the inequalities (7.28) and (7.30) arising from the constraint of asymptotic freedom for the SU(N) and SU(2) gauge interactions, respectively, and the requirement that the theory must not have any global SU(2) anomaly, Eq. (7.26). The notation $12 \le N_{even} \le 20$ denotes the even values of N in this range. The notation $13 \le N_{odd} \le 21$ denotes the odd values of N in this range. For $N \ge 22$, the inequality (7.30) has only the trivial solution $p = 0$ for which the theory is a pure gauge theory with no fermions and hence is not a chiral gauge theory.

N	allowed values of p
3	$p = 2,\ 4,\ 6$
4	$1 \le p \le 5$
5	$p = 2,\ 4$
6	$1 \le p \le 3$
7	$p = 2$
8	$p = 1,\ 2$
9	$p = 2$
10	$p = 1,\ 2$
11	no sol. with $p \ne 0$
$12 \le N_{even} \le 20$	$p = 1$
$13 \le N_{odd} \le 21$	no sol. with $p \ne 0$
$N \ge 22$	no sol. with $p \ne 0$

The allowed values of N and p are thus the integers $N \geq 3$ and $p \geq 1$ in this allowed region that satisfy the conditions (7.28), (7.30), and (7.26). We list these in Table 7.2. Several comments are in order concerning these allowed values of N and p. First, as N increases through the value $N = 22$, the maximum value of p allowed by the inequality (7.30) decreases below 1, so that for $N > 22$, this inequality (7.30) has only the trivial (integral) solution $p = 0$ for which the theory is a pure gauge theory with no fermions and hence not of interest here. Second, for odd N, one sees that the condition (7.26) for the theory to be free from a global SU(2) anomaly restricts p to even values.

We next analyze the UV to IR evolution and gauge symmetry breaking in this model. If the SU(N) gauge interaction is sufficiently strong and if it dominates over the SU(2) gauge interaction, then this SU(N) interaction forms bilinear fermion condensates that break the SU(2) gauge symmetry. We denote the scale at which this occurs as Λ. As regards the SU(N) gauge interaction, the most attractive channel for fermion condensation is

$$\mathrm{SU}(N): \quad \Box \times \bar{\Box} \to 1, \tag{7.31}$$

in terms of Young tableaux, or equivalently, $N \times \bar{N} \to 1$, in terms of the dimensionalities of the SU(N) representations, with associated condensates

$$\langle \sum_{a=1}^{N} \psi_{i,L}^{a,\alpha\ T} C \chi_{a,j,L} \rangle \ , \tag{7.32}$$

where $i \in \{1, ...p\}$ and $j \in \{1, ..., 2p\}$. (Here and below, when a condensate is given, it is understood that the hermitian conjugate condensate is also present.) This channel has

$$\Delta C_2 = 2C_2(\Box) = \frac{N^2 - 1}{N} \ . \tag{7.33}$$

Each of the condensates in Eq. (7.32) breaks the SU(2) gauge symmetry completely (and is invariant under the SU(N) gauge symmetry, as is clear from (7.31)). The fermions involved in these condensates, and the SU(2) gauge bosons, gain dynamical masses of order Λ.

If, on the other hand, the SU(2) interaction is sufficiently strong and if it dominates over the SU(N) interaction, then this SU(2) interaction produces bilinear fermion condensates in the most attractive SU(2) channel $2 \times 2 \to 1$, with associated condensates of the form

$$\langle \epsilon_{\alpha\beta} \psi_{i,L}^{a,\alpha\ T} C \psi_{j,L}^{b,\beta} \rangle \ . \tag{7.34}$$

We denote the scale where this occurs as Λ'. The attractiveness measure for condensate formation in this channel is $\Delta C_2 = 2C_2(\Box) = 3/2$. From the general symmetry property (7.6), it follows that if, as in Eq. (7.34), one contracts the SU(2) gauge indices α and β antisymmetrically via the SU(2) $\epsilon_{\alpha\beta}$ tensor, then the combination of SU(N) and generational indices is antisymmetric. That is, in the operator product (7.34), either the SU(N) gauge indices are antisymmetric and the generational indices are symmetric, so the condensate is proportional to

$$\begin{aligned} &\langle \epsilon_{\alpha\beta}(\psi^{a,\alpha}_{i,L}{}^T C \psi^{b,\beta}_{j,L} - \psi^{b,\alpha}_{i,L}{}^T C \psi^{a,\beta}_{j,L} \\ + \quad & \psi^{a,\alpha}_{j,L}{}^T C \psi^{b,\beta}_{i,L} - \psi^{b,\alpha}_{j,L}{}^T C \psi^{a,\beta}_{i,L})\rangle \end{aligned} \tag{7.35}$$

or the SU(N) gauge indices are symmetric and the generational indices are antisymmetric, so the condensate is proportional to

$$\begin{aligned} &\langle \epsilon_{\alpha\beta}(\psi^{a,\alpha}_{i,L}{}^T C \psi^{b,\beta}_{j,L} + \psi^{b,\alpha}_{i,L}{}^T C \psi^{a,\beta}_{j,L} \\ - \quad & \psi^{a,\alpha}_{j,L}{}^T C \psi^{b,\beta}_{i,L} - \psi^{b,\alpha}_{j,L}{}^T C \psi^{a,\beta}_{i,L})\rangle \ . \end{aligned} \tag{7.36}$$

The SU(N) gauge interaction, although assumed to be weaker than the SU(2) gauge interaction, is not assumed to be negligible, and it prefers the condensation channel that is the MAC as regards SU(N). Now

$$\Delta C_2 = \frac{N+1}{N} \quad \text{for} \quad \Box \times \Box \to \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \tag{7.37}$$

whereas

$$\Delta C_2 = -\frac{N-1}{N} \quad \text{for} \quad \Box \times \Box \to \Box\Box \ , \tag{7.38}$$

so the $\Box \times \Box \to \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ channel is the MAC, and indeed, the $\Box \times \Box \to \Box\Box$ channel is repulsive. Therefore, we conclude that in this case where SU(2) is more strongly coupled than SU(N), the expected condensation channel is, in an obvious notation,

$$(\Box, 2) \times (\Box, 2) \to (\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, 1) \tag{7.39}$$

with associated condensate (7.35). This condensate, which is of the form $\langle T^{[ab]}\rangle$, where $T^{[ab]}$ is a rank-2 antisymmetric tensor of SU(N), breaks SU(N) as follows :

$$\langle T^{[ab]}\rangle : \ \mathrm{SU}(N) \to H = \begin{cases} \mathrm{SU}(2) & \text{if } N = 3 \\ \mathrm{SU}(N-2) \otimes \mathrm{SU}(2) & \text{if } N \ge 4 \end{cases} \ . \tag{7.40}$$

The fermions involved in the condensate and the gauge bosons in the coset $\mathrm{SU}(N)/H$ gain dynamical masses of order Λ' and are integrated out of the low-energy effective field theory that is operative as the reference scale μ decreases below Λ'. The fermion condensates that form in both the strong-SU(N) and strong-SU(2) situations also break global flavor symmetries. S-ince we have already analyzed this sort of global flavor symmetry breaking in our previous works [1, 22], we will not pursue this here, instead focusing on the gauge symmetry breaking.

7.4.2 Model B

This model, denoted Model B, has the same gauge group as Model A, but has an enlarged chiral fermion sector which also contains $p_1 \equiv p'$ copies of the SU(N)-singlet, SU(2)-doublet fermion

$$\eta^{\alpha}_{j,L} \ , \ j = 1, ..., p' \ : \quad p'(1,2) \ . \tag{7.41}$$

Thus, the fermion content of Model B can be categorized as being of the form

$$\{f_{ns,ns}, \ f_{ns,s}, \ f_{s,ns}\} \tag{7.42}$$

in the notation of Eq. (7.24). Depending on the value of p', these additional fermions may have gauge-invariant bare mass terms of the form

$$\epsilon_{\alpha\beta}\eta^{\alpha\ T}_{i,L} C \eta^{\beta}_{j,L} \ , \tag{7.43}$$

where $i \neq j$ and $1 \leq i, j \leq p'$. Using the general symmetry property (7.6) and taking account of the antisymmetric contraction of the SU(2) gauge indices α and β with the $\epsilon_{\alpha\beta}$ tensor, it follows that the fermion operator in (7.43) is automatically antisymmetrized in the flavor indices i and j, so if $p' = 1$, then it vanishes identically. If $p' \geq 2$, then the $\{f_{s,ns}\}$ fermions constitute a vectorlike subsector in the full chiral gauge theory.

The sector of SU(N)-nonsinglet fields in Model B is the same as in Model A, so the SU(N) gauge interaction is again vectorial and hence is free from any gauge anomaly, as is the SU(2) gauge interaction. The condition that the SU(2) part of the theory should be free of any global anomaly is that the number of SU(2) doublets, denoted N_d, is even, i.e.,

$$N_d = pN + p' \text{ is even,} \tag{7.44}$$

and we require that this condition be satisfied.

The one-loop coefficient of the SU(N) beta function, $b_{1,\mathrm{SU}(N)}$, is given by (7.27), as in Model A, so p is subject to the same upper bound from the requirement that the SU(N) interaction must be asymptotically free, namely (7.28).

The one-loop coefficient of the SU(2) beta function is

$$b_{1,\mathrm{SU}(2)_L} = \frac{1}{3}[22 - (pN + p')] \ , \tag{7.45}$$

so the requirement that the SU(2) gauge interaction should be asymptotically free implies that

$$pN + p' < 22 \ . \tag{7.46}$$

The allowed values of N, p, and p' for Model B are thus the integers $N \geq 3$, $p \geq 1$, and $p' \geq 1$ satisfying the conditions (7.28), (7.46)), and (7.44). There are too many values to list in a table analogous to Table 7.2, but we mention that for $N = 3$, the allowed values of (p, p') are $(1, 2k+1)$ with $0 \leq k \leq 8$; $(2, 2k)$ with $1 \leq k \leq 7$; $(3, 2k+1)$ with $0 \leq k \leq 5$; $(4, 2k)$ with $1 \leq k \leq 4$; $(5, 2k+1)$ with $0 \leq k \leq 2$; and the single pair $(6, 2)$. As in Model A, as N increases, the allowed set of values of p and p' is progressively reduced, and for sufficiently large N, there are no nontrivial solutions to the three conditions. For example, for $N = 16$, there are only two allowed sets of (p, p'), namely (1,2) and (1,4); for $N = 17$, there are again two sets, namely (1,1) and (1,3), while for $N = 18$, there is only one, (1,2), and for $N = 19$, there is only one, (1,1). For $N \geq 20$, there are no allowed (nonzero) values of p and p' in this model.

Since Model B is the same as Model A as regards the SU(N)-nonsinglet fermion content, it follows that if the SU(N) gauge interaction is sufficiently strong and dominates over the SU(2) interaction, then the resultant bilinear fermion condensate formation is the same as in Model A.

However, if the opposite is the case, i.e., if the SU(2) gauge interaction is strong enough and dominates over the SU(N) interaction, then, depending on the value of p', two additional type of fermion condensates may be produced. These all have the same SU(2) attractiveness measure, as given before, namely, $\Delta C_2 = 3/2$ and hence, if SU(N) interactions are negligible, they are expected to form at essentially the same Euclidean scale, which we again denote as Λ. Thus, in addition to the condensate(s) (7.35), the SU(2) gauge interaction can lead to condensation in the channel

$$(1, 2) \times (1, 2) \to (1, 1) \tag{7.47}$$

with the associated condensate(s)

$$\langle \epsilon_{\alpha\beta} \eta^{\alpha}_{i,L}{}^{T} C \eta^{\beta}_{j,L} \rangle \ , \tag{7.48}$$

where $1 \le i, j \le p'$. From (7.6), it follows that the bilinear fermion operator product in (7.48) is antisymmetric in the copy indices i and j and hence vanishes identically if $p' = 1$. As is evident from (7.47), this condensate (7.48) preserves the full $\mathrm{SU}(N) \otimes \mathrm{SU}(2)_L$ gauge symmetry. The fermions involved in these condensates gain dynamical masses of order the condensation scale, denoted Λ, and are integrated out in the low-energy effective field theory that is operative as the reference scale μ decreases below Λ.

The second possible additional condensation channel is

$$(N, 2) \times (1, 2) \to (N, 1) \tag{7.49}$$

with the associated condensate(s)

$$\langle \epsilon_{\alpha\beta} \psi^{a,\alpha}_{i,L}{}^{T} C \eta^{\beta}_{j,L} \rangle \ , \tag{7.50}$$

where $1 \le i \le p$ and $1 \le j \le p'$. Consider the condensates (7.50) with a given i, say $i = 1$. This set of condensates (7.50) breaks $\mathrm{SU}(N)$ to $\mathrm{SU}(N - p')$ if $1 \le p' \le N - 2$ and and breaks $\mathrm{SU}(N)$ completely if $p' \ge N - 1$. To show this, note that without loss of generality we may pick $a = N$ and $j = 1$ for one of these condensates. This condensate, $\langle \epsilon_{\alpha\beta} \psi^{N,\alpha}_{1,L}{}^{T} C \eta^{\beta}_{1,L} \rangle$, breaks $\mathrm{SU}(N)$ to the subgroup $\mathrm{SU}(N - 1)$. The fermions $\psi^{N,\alpha}_{1,L}$ and $\eta^{\beta}_{1,L}$ involved in this condensate gain dynamical masses of order the scale at which this condensate forms. Next, consider the condensate of the form (7.50), where now only the $\mathrm{SU}(N-1)$ gauge indices $a \in \{1, ..., N-1\}$ are dynamical. Again, by convention, we may pick the $\mathrm{SU}(N - 1)$ gauge index in this condensate to be $N - 1$ and the copy index on the $\eta^{\beta}_{j,L}$ fermion to be $j = 2$. This breaks $\mathrm{SU}(N-1)$ to $\mathrm{SU}(N-2)$ and the fermions $\psi^{N-1,\alpha}_{1,L}$ and $\eta^{\beta}_{2,L}$ involved in this condensate gain dynamical masses of order the condensation scale. This process continues until $\mathrm{SU}(N)$ is broken to $\mathrm{SU}(N - p')$ if $N - p' \ge 2$ or until $\mathrm{SU}(N)$ is completely broken if $N - p' \le 1$. A vacuum alignment argument suggest that it is plausible that this pattern of breaking would also hold for other values $i = 2, ..., p$. As noted above, since the SU(2) attractiveness measure of all of these condensates, $\Delta C_2 = 3/2$ is the same, one expects that they form at essentially the same scale.

7.5 $\mathrm{SO}(4k+2) \otimes \mathrm{SU}(2)$ Theories

It is also of interest to study chiral gauge theories with direct-product groups group that involve a safe $\mathrm{SO}(N)$ group. We recall that if N is odd or if N is even and $N = 4k$, $k \geq 1$, then $\mathrm{SO}(N)$ has only real representations, while if $N = 4k+2$ with $k \geq 2$, then the theory has complex representations but is safe (i.e., has no anomaly for any representation) [106, 114]. With this motivation, we consider chiral gauge theories with the gauge group

$$G = \mathrm{SO}(4k+2) \otimes \mathrm{SU}(2) \quad \text{with } k \geq 2 \ . \tag{7.51}$$

These are of the form (cs, cs) in the general classification given in Section 7.1. Since N is even, it is also convenient to introduce an integer $r = N/2$:

$$N = 4k+2 = 2r \ , \quad k \geq 2 \ , \tag{7.52}$$

so $r = 2k+1$. As before, we write all fermions as left-handed. We start by considering the general fermion content

$$\sum_{\mathcal{R},\ \mathcal{R}'} \Big[n_{\mathcal{R}}\, (\mathcal{R}, 1) + p\, (\mathcal{R}', \Box) \Big] \ , \tag{7.53}$$

where $\mathcal{R}$ and $\mathcal{R}'$ are representations of $\mathrm{SO}(4k+2)$. We include only complex $\mathcal{R}$ and $\mathcal{R}'$ since the use of a real $\mathcal{R}$ or $\mathcal{R}'$ would lead to a vectorlike subsector, so the model would not be irreducibly chiral.

Using the relevant group invariants, we calculate the one-loop term in the beta function for the $\mathrm{SO}(N)$ gauge coupling with N given by (7.52) to be

$$b_{\mathrm{SO}(4k+2),1} = \frac{2}{3}\Big[11(r-1) - \sum_f \Big(n_{\mathcal{R}} T_{\mathcal{R}} + 2 p_{\mathcal{R}'} T_{\mathcal{R}'} \Big) \Big] \ . \tag{7.54}$$

We calculate the one-loop term in the $\mathrm{SU}(2)$ beta function to be

$$b_{\mathrm{SU}(2),1} = \frac{1}{3}\Big[22 - 2 \sum_{\mathcal{R}'} p_{\mathcal{R}'} \mathrm{dim}(\mathcal{R}') \Big] \ . \tag{7.55}$$

Because the first terms in square brackets in Eq. (7.54) and (7.55) are, respectively, linear in r and a constant, while the relevant $T_{\mathcal{R}}$, $T_{\mathcal{R}'}$, and $\mathrm{dim}(\mathcal{R}')$ grow exponentially rapidly with r, the asymptotic freedom of the $\mathrm{SO}(2r)$ and $\mathrm{SU}(2)$ gauge interactions places strong restrictions on the fermion content and the value of N. For our purposes, it will be sufficient to consider the simplest models of this type, with (complex) $\mathcal{R} = \mathcal{R}'$. We will consider three specific models, which we label Models A, B, and C.

7.5.1 Model A

We first briefly consider the case where the fermion sector has the form $\{f_{ns,s}\}$, i.e, all of the fermions are singlets under SU(2). In this case, the gauge group effectively reduces to SO(N), with N given by (7.52). We choose the minimal complex representation for the fermions, namely the spinor representation, denoted $\mathcal{S}$, of dimension $\dim(\mathcal{S}) = 2^{r-1} = 2^{2k}$ (see Appendix A) and include n copies of these, so the fermion content is

$$\omega_{i,L}, \quad i = 1.,,,n \ : \quad n\,(\mathcal{S}, 1) \ , \tag{7.56}$$

where the first and second entries in the parentheses here and below are the representations of SO(N) and SU(2), respectively. The general formula for the one-loop term in the beta function for the SO(N) gauge coupling, Eq. (7.54) for this Model A reduces to

$$b_{\mathrm{SO}(2r),1} = \frac{2}{3}\Big[11(r-1) - 2^{r-4}n\Big] \ . \tag{7.57}$$

The requirement that the SO(N) gauge interaction should be asymptotically free implies that

$$n < \frac{11(r-1)}{2^{r-4}} \ . \tag{7.58}$$

This has only a finite number of solutions for n that are nontrivial, i.e., have $n \ge 1$, and, indeed, also a finite number of solutions for r.

$$G_1 = \mathrm{SO}(10) \ (i.e., \ k=2, \ r=5) \ \Rightarrow n \le 21 \tag{7.59}$$

$$G_1 = \mathrm{SO}(14) \ (i.e., \ k=3, \ r=7) \ \Rightarrow n \le 8 \tag{7.60}$$

$$G_1 = \mathrm{SO}(18) \ (i.e., \ k=4, \ r=9) \ \Rightarrow n \le 2 \tag{7.61}$$

For $k \ge 5$, i.e., $r \ge 11$, the upper bound on n is less than unity, precluding any fermions.

We assume some initial value of the SO($2r$) gauge coupling in the deep UV and then evolve the theory downward in Euclidean scale μ. Recall that the direct product of two spinor representations of SO(N) with N given by (7.52) is [104]

$$\mathcal{S} \times \mathcal{S} = 2^{2k} \times 2^{2k} = \sum_{\ell=0}^{k-1} A_{2\ell+1} + R_{2k+1;2} \ , \tag{7.62}$$

where A_t denotes the rank-t antisymmetric tensor representation and R_{2k+1} is a certain self-dual representation. The symmetry of the A_t with respect to the interchange of the two spinor representations in the direct product is given by $(-1)^{u(r,t)}$, where $u(r,t) = (r-t)(r-t-1)/2$ [104]. Thus, for example, one has, for the lowest relevant value of k, namely $k = 2$, i.e., $G_1 = \mathrm{SO}(10)$,

$$\begin{aligned} \mathrm{SO}(10): \quad \mathcal{S} \times \mathcal{S} &= 2^4 \times 2^4 = A_1 + A_3 + R_{5;2} \\ &= 10_s + 120_a + 126_s \ , \end{aligned} \tag{7.63}$$

where the subscripts s and a denote the symmetric and antisymmetric property of these representations under interchange of the spinors in the direct product. In general, for $\mathrm{SO}(2k+2)$, from the form of $u(r,t)$, it follows that A_1 is symmetric (resp. antisymmetric) under interchange of the spinors in the direct product for even k (resp. odd k), while A_3 is antisymmetric (resp. symmetric) under interchange of these spinors for even k (resp. odd k).

Assuming that the $\mathrm{SO}(N)$ coupling becomes strong enough to produce a bilinear fermion condensate, the MAC is

$$\mathrm{SO}(N)\ \mathrm{MAC}: \quad \mathcal{S} \times \mathcal{S} \to \square \ , \tag{7.64}$$

with attractiveness measure

$$\begin{aligned} \Delta C_2 &= 2C_2(\mathcal{S}) - C_2(\square) = \frac{(N-1)(N-4)}{8} \\ &= \frac{(2r-1)(r-2)}{4} = \frac{(4k+1)(2k-1)}{4} \ . \end{aligned} \tag{7.65}$$

Since $r \ge 5$, i.e., $k \ge 2$, this is always positive. The associated condensate is $\langle \omega_{i,L}^T C \omega_{j,L} \rangle$, where $1 \le i,j \le n$. From the general result (7.6), it follows that the bilinear fermion operator $\omega_{i,L}^T C \omega_{j,L}$ in this condensate is (i) symmetric under interchange of spinors in the $\mathcal{S} \times \mathcal{S}$ direct product in (7.64) and hence symmetric in the flavor indices i,j if k is even; (ii) antisymmetric under interchange of spinors and hence antisymmetric in the flavor indices i,j if k is odd. Therefore, explicitly,

$$k \text{ even} \ \Rightarrow \ \langle \omega_{i,L}^T C \omega_{j,L} + \omega_{j,L}^T C \omega_{i,L} \rangle \ , \quad 1 \le i,j \le n \tag{7.66}$$

and

$$k \text{ odd} \ \Rightarrow \ \langle \omega_{i,L}^T C \omega_{j,L} - \omega_{j,L}^T C \omega_{i,L} \rangle \ , \quad 1 \le i,j \le n \ . \tag{7.67}$$

In both cases, if this condensate forms, then, since it transforms as the fundamental (vector) representation of the gauge group $SO(4k+2)$, it breaks this symmetry to $SO(4k+1)$, which is vectorial and does not break further.

However, if $n = 1$ and k is odd, e.g., for SO(14) (i.e., $k = 3$), then this condensate in the MAC channel vanishes identically. In this case, we consider the next channel in Eq. (7.62), namely

$$\mathcal{S} \times \mathcal{S} \to A_3 \tag{7.68}$$

with attractiveness measure

$$\begin{aligned} \Delta C_2 &= 2C_2(\mathcal{S}) - C_2(A_3) = \frac{(N-4)(N-9)}{8} \\ &= \frac{(r-2)(2r-9)}{4} = \frac{(2k-1)(4k-7)}{4} \ . \end{aligned} \tag{7.69}$$

For the relevant value of k, namely $k = 3$, this is $\Delta C_2 = 25/4$.

7.5.2 Model B

Here we consider a model with the gauge group (7.51) with (7.52) and fermion content of the form $\{f_{ns,ns}\}$, namely

$$\psi^{\alpha}_{i,L}, \ i = 1, ..., p : \quad p\,(\mathcal{S}, \square) \ . \tag{7.70}$$

We denote this as Model B. Since there are an even number of SU(2) doublets, this theory has no global $SU(2)_L$ anomaly.

The general formulas for the one-loop coefficients in the $SO(N)$ beta function (with N given by (7.52)) and in the SU(2) beta function displayed in Eqs. (7.54) and (7.55) reduce, for this Model B, to

$$b_{SO(2r),1} = \frac{2}{3}[11(r-1) - 2^{r-3}p] \tag{7.71}$$

and

$$b_{1,SU(2)_L} = \frac{2}{3}(11 - 2^{r-2}p) \ . \tag{7.72}$$

Hence, the respective conditions that the $SO(2r)$ and SU(2) gauge interactions should be asymptotically free are

$$p < \frac{11(r-1)}{2^{r-3}} \tag{7.73}$$

and

$$p < \frac{11}{2^{r-2}} \ . \tag{7.74}$$

Since we take $k \ge 2$, i.e., $r \ge 5$, for our theories, the only possible nontrivial value for p allowed by the constraint (7.74) is $p = 1$ and, furthermore, this is only possible for the lowest value of k, namely $k = 2$, and thus $G_1 = \mathrm{SO}(10)$. No SO($4k+2$) theories of this Model B type with nonzero fermion content are allowed by the asymptotic freedom constraint if $k \ge 3$.

We note that there is consequently no (continuous) nonanomalous global flavor symmetry of the Lagrangian for this theory. Since there is only one copy of the $(\mathcal{S}, \Box)$ fermion $\psi^\alpha_{i,L}$, we shall henceforth drop the flavor index and write this field simply as ψ^α_L.

If the SO(10) gauge interaction is sufficiently strong and dominates over the SU(2) gauge interaction, then it produces a condensate in the SO(10) MAC, (7.64), thereby breaking the SO(10) gauge symmetry to SO(9), which is vectorial and does not break further. The condensate is $\langle \psi^{\alpha\ T}_L C \psi^\beta_L \rangle$. As noted above in Section 7.5.1, for SO($4k+2$), the $\Box = A_1$ that occurs in the Clebsch-Gordan decomposition of the direct product $\mathcal{S} \times \mathcal{S}$ in (7.64) is symmetric (resp. antisymmetric) under interchange of these spinors if k is even (resp. odd). Since $k = 2$ is even here, it follows that this $\Box$ representation is symmetric under interchange of the spinors in the direct product. From the property (7.6), it then follows that the SU(2) gauge indices must also be symmetric, i.e., the SU(2) channel is $2 \times 2 \to 3_s$, so the operator product transforms as the adjoint (equivalently, the rank-2 symmetric tensor) representation of SU(2) and hence can be written as proportional to

$$\langle \psi^{\alpha\ T}_L C \psi^\beta_L + \psi^{\beta\ T}_L C \psi^\alpha_L \rangle \ . \tag{7.75}$$

Hence, including both factor groups, in this case of a strong and dominant SO(10) gauge interaction with even k (viz., $k = 2$), the condensation is in the channel

$$k \text{ even} \ \Rightarrow \ (\mathcal{S}, \Box) \times (\mathcal{S}, \Box) \to (\Box_s, adj_s) = ((4k+2)_s, 3_s) \ . \tag{7.76}$$

In addition to breaking SO(10) to SO(9), this condensate SU(2) to a subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$.

The $2 \times 2 \to 3_s$ channel is actually a repulsive channel for the SU(2) interaction, with $\Delta C_2 = -1/2$. If the SU(2) gauge interaction is weak enough, this does not matter, but if it is moderately strong, although weaker than the SO(10) gauge interaction, it might prevent the condensate from forming. However, we assume that the SO(10) coupling is sufficiently strong at a given scale μ so that this condensate does form.

Having analyzed the situation in which the SO(10) gauge coupling is strong and dominates over the SU(2) gauge coupling, we next analyze the opposite situation in which the SU(2) gauge coupling becomes sufficiently strong and dominates over the SO(10) coupling. The condensate then forms in the MAC for SU(2), which is $2 \times 2 \to 1_a$, involving an antisymmetric contraction of SU(2) indices with the $\epsilon_{\alpha\beta}$ tensor.

$$\langle \epsilon_{\alpha\beta} \psi_L^{\alpha\ T} C \psi_L^{\beta} \rangle \ . \tag{7.77}$$

The general result (7.6) then implies that the relevant representation in the Clebsch-Gordan decomposition of the direct product $\mathcal{S} \times \mathcal{S}$ is antisymmetric, and we therefore denote it as R_a. As discussed above, given that k is even here, the representation that would normally be favored as the MAC in the direct product of two spinors, (7.62), namely the $\Box$ representation, is symmetric rather than antisymmetric, and hence R_a cannot be $\Box$. Instead, the lowest-dimension representation in the expansion (7.62) that is odd under interchange of the spinors is A_3 with dimension $\binom{4k+2}{3}$, so the condensation channel is

$$(\mathcal{S}, \Box) \times (\mathcal{S}, \Box) \to ((A_3)_a, 1_a) \ . \tag{7.78}$$

The measure of attractiveness of this channel is given by the ΔC_2 in Eq. (7.69) and is always positive for $k \geq 2$. Explicitly, for our SO(10) Model B theory, the A_3 representation has dimension 120. When expressed as a sum of product representations of various SO(10) subgroups, the 120-dimensional representation has no singlets under either of the maximal (i.e., rank-5) subgroups SU(5) $\otimes$ U(1) and SU(4) $\otimes$ SU(2) $\otimes$ SU(2), or the rank-4 subgroup SO(9), but does contain a singlet under the rank-4 subgroup SO(7) $\otimes$ SU(2) [104]. It therefore breaks SO(10) to SO(7) $\otimes$ SU(2).

7.5.3 Model C

Here we analyze a model, denoted Model C, that has a fermion sector which is a combination of the fermion sectors of Model A in Section 7.5.1 and Model B in Section 7.5.2, and thus is of the form $\{f_{ns,s},\ f_{ns,ns}\}$. These fermions consist of n copies of the $(\mathcal{S}, 1)$ fermion $\omega_{i,L}$, $i = 1, ..., n$, as in Eq. (7.56) and a single copy of the $(\mathcal{S}, \Box)$ fermion, $\psi_{1,L}^{\alpha}$, as in Eq. (7.70).

The one-loop coefficient in the beta function of the SU(2) gauge interaction in this Model C is the same as (7.72) for Model B, and hence the requirement that the SU(2) gauge interaction must be asymptotically free restricts $p \le 1$. The case $p = 0$ reduces to Model A, which we have already discussed. Therefore, as indicated, we take $p = 1$ here. This, in turn, restricts k to be equal to 2, i.e., $G_1 = \mathrm{SO}(10)$.

The one-loop coefficient in the SO(10) beta function for this Model C is

$$b_{1,\mathrm{SO}(10)} = \frac{2}{3}(20 - n) \ , \tag{7.79}$$

so the asymptotic freedom of the SO(10) gauge interaction implies that $n < 20$.

If the SO(10) gauge interaction is sufficiently strong and dominates over the SU(2) interaction, then the resultant condensates include those analyzed for Models A and B above, together with a new type of condensate. This new condensate occurs in the channel

$$(\mathcal{S}, 1) \times (\mathcal{S}, \Box) \to (\Box, \Box) \tag{7.80}$$

with corresponding condensate

$$\langle \omega_{i,L}^T C \psi_L^\alpha \rangle \ , \quad i \in \{1, ..., n\} \ . \tag{7.81}$$

This condensate breaks SO(10) to SO(9), which is vectorial and does not break further.

If, on the other hand, the SU(2) gauge interaction is sufficiently strong and dominates over the SO(10) interaction, then the condensate formation and symmetry-breaking is the same as for Model B, discussed in Section 7.5.2.

7.6 $\mathrm{SO}(4k+2) \otimes \mathrm{SU}(M)$ Theory

Here we consider a chiral gauge theory with the gauge group

$$\mathrm{SO}(N) \otimes \mathrm{SU}(M) \ , \text{with } N = 4k + 2 = 2r \ \text{ and } M \ge 3 \ . \tag{7.82}$$

We will show that the constraint of asymptotic freedom of both gauge interactions limits k to the single value $k = 2$, but in order to show this, we must first keep $k \geq 2$ general. The fermion content is the sum over representations $\mathcal{R}$ of

$$\begin{aligned} & \dim(\mathcal{R}_{\mathrm{SU}(M)})\,(\mathcal{R}_{\mathrm{SO}(4k+2)},1) + (\bar{\mathcal{R}}_{\mathrm{SO}(4k+2)},\bar{\mathcal{R}}_{\mathrm{SU}(M)}) \\ + & \dim(\mathcal{R}_{\mathrm{SO}(4k+2)})\,(1,\mathcal{R}_{\mathrm{SU}(M)}) \ . \end{aligned} \tag{7.83}$$

In the classification of Section 7.1), this theory is of the (cs, cav) type. We take $M \geq 3$ since the theory with $M = 2$ has a vectorlike subsector comprised of the $(1, 2)$ fermions and is therefore not irreducibly chiral. Note that even if $M = 2$, this theory does not coincide with any of Models A, B, or C in Section 7.5 because those models also avoided $(1, \Box) = (1, 2)$ fermions that would have constituted a vectorlike subsector. However, if one were to take $M = 2$, then the SO$(4k + 2)$-nonsinglet fermion sector would coincide with that of Model B in Section 7.5. We will show below that M is limited to a finite set of values by the constraint of asymptotic freedom. For our present purposes, it will suffice to consider the simplest realization of this theory, with a single representation $\mathcal{R}$ of SO$(4k + 2)$, namely the smallest complex one, the spinor, and the smallest nonsinglet representation of SU(2), namely the fundamental. The resultant fermion content is thus

$$p\,(\mathcal{S},\Box)\ , \quad 2^{r-1}p\,(1,\bar{\Box})\ . \tag{7.84}$$

The one-loop coefficient of the SO$(4k+2)$ beta function (with $4k+2 = 2r$) is

$$b_{\mathrm{SO}(4k+2)} = \frac{2}{3}\Big[11(r-1) - 2^{r-4}pM\Big]\ . \tag{7.85}$$

The requirement that the SO$(4k + 2)$ gauge interaction must be asymptotically free then yields the upper bound

$$p < \frac{11(r-1)}{2^{r-4}M}\ . \tag{7.86}$$

Although we restrict $M \geq 3$, we note that if one were to take $M = 2$, then this would be the same as the upper bound (7.73) on p for Model B in Section 7.5). The fact that we take $M \geq 3$ here makes this a more stringent upper bound than (7.73).

We denote the fermion fields for this theory as

$$\psi^{\alpha}_{i,L}\ , \quad i = 1, ..., p: \quad p\,(\mathcal{S},\Box) \tag{7.87}$$

and

$$\eta_{\alpha,j,L}\ ,\, j=1,...,2^{r-1}p: \quad 2^{r-1}p\,(1,\overline{\Box})\ , \tag{7.88}$$

where α is an SU(M) gauge index and i,j are flavor indices.

The one-loop coefficient of the SU(M) beta function is

$$b_{\rm SU(\mathit{M})} = \frac{1}{3}(11M - 2^r p)\ . \tag{7.89}$$

The requirement that the SU(M) gauge interaction must be asymptotically free then yields the upper bound

$$p < \frac{11M}{2^r}\ . \tag{7.90}$$

For the relevant range $M \geq 3$, these two asymptotic freedom constraints can only be satisfied for r equal to its minimal value, $r = 5$, i.e., $k = 2$ and $G_1 = {\rm SO(10)}$; furthermore, given that $k = 2$, there are only a finite set of pairs (M,p) that satisfy the constraints. For the two integer intervals $3 \leq M \leq 5$ and $11 \leq M \leq 21$, only the value $p = 1$ is allowed, while for $6 \leq M \leq 10$, p may take on the values 1 or 2. If $M \geq 22$, there are no allowed solutions for p. Our general construction is thus reduced to the finite family of chiral gauge theories with the gauge groups SO(10) $\otimes$ SU(M) with $3 \leq M \leq 21$ and the aforementioned possible values of p as a function of M.

If the SO(10) gauge coupling becomes sufficiently large and dominates over the SU(M) gauge coupling, then the former can produce condensation in the SO(10) MAC, namely (7.64). Since the $\Box$ is symmetric under interchange of the spinors in (7.64) for even k and hence, in particular, for $k = 2$, i.e., SO(10), it follows from our general result (7.6) that the combination of the SU(M) and flavor product S_{ij} must be symmetric. For the values of M, namely $3 \leq M \leq 5$ and $11 \leq M \leq 21$ that allow only $p = 1$, it follows that the flavor product must be symmetric, as $S_{ij} = S_{11}$ and hence that the channel is, in terms of the full representations,

$$(\mathcal{S},\Box) \times (\mathcal{S},\Box) \to (\Box_s, \Box\Box) \tag{7.91}$$

with the condensate

$$\langle \psi^{\alpha\ T}_{1,L} C \psi^{\beta}_{1,L} \rangle\ . \tag{7.92}$$

The SO(10) ΔC_2 measure of attractiveness for this channel is given by the $N = 10$ special case of Eq. (7.65), namely 27/4. However, the SU(M) ΔC_2

value is negative, as is evident from Eq. (7.38), setting $M = N$, so this is a repulsive channel as regards the SU(M) interaction. This breaks SO(10) to SO(9), which is vectorial, and does not break further. Using a vacuum alignment argument, one may infer that $\alpha = \beta$ so that the condensate (7.92) breaks SU(M) to SU($M-1$).

For the interval $6 \le M \le 10$ where the theory allows $p = 2$, the dynamics could instead produce a condensate in the channel

$$(\mathcal{S}, \square) \times (\mathcal{S}, \square) \to (\square_s, \begin{smallmatrix}\square\\\square\end{smallmatrix}) \ , \tag{7.93}$$

where the flavor product S_{ij} is antisymmetric, so that the condensate is

$$\langle \psi^{\alpha\ T}_{1,L} C \psi^{\beta}_{2,L} - \psi^{\alpha\ T}_{2,L} C \psi^{\beta}_{1,L} \rangle \ . \tag{7.94}$$

In addition to being attractive as regards the SO(10) interaction, the channel (7.93) is also attractive with respect to the SU(M) interaction, with ΔC_2 given by Eq. (7.37) with $N = M$. Hence, for M in the interval $6 \le M \le 10$ where $p = 2$ is allowed, we infer that the preferred condensation channel in the case where SO(10) is strong is (7.93). This breaks SO(10) to SO(9) and SU(M) to SU($M-2$) $\otimes$ SU(2).

7.7 $\mathrm{SO}(4k+2) \otimes \mathrm{SO}(4k'+2)$ Theory

Here we explore a chiral gauge group of the (cs, cs) type, in our classification from Section (7.1). For this purpose, we choose the gauge group

$$\mathrm{SO}(4k+2) \otimes \mathrm{SO}(4k'+2) \ , \quad \text{where } k, \ k' \ge 2 \tag{7.95}$$

and fermion content consisting of p copies of the bi-spinor representation, $(\mathcal{S}, \mathcal{S})$. We set $N = 4k + 2 = 2r$ and $N' = 4k' + 2 = 2r'$. Although this family of theories ostensibly depends on the three parameters k, k', and p, we will show that there is only one allowed choice for these three parameters.

The one-loop coefficients in the SO($4k+2$) and SO($4k'+2$) beta functions are

$$b_{\mathrm{SO}(4k+2),1} = \frac{2}{3}\Big[11(r-1) - 2^{r+r'-5} p\Big] \tag{7.96}$$

and

$$b_{\mathrm{SO}(4k'+2),1} = \frac{2}{3}\Big[11(r'-1) - 2^{r+r'-5} p\Big] \ . \tag{7.97}$$

The requirements that the SO($4k+2$) and SO($4k'+2$) gauge interactions must be asymptotically free yield the upper bounds

$$p < \frac{11(r-1)}{2^{r+r'-5}} \tag{7.98}$$

and

$$p < \frac{11(r'-1)}{2^{r+r'-5}} \ . \tag{7.99}$$

These can only be satisfied by the single set of values $r = r' = 5$ and $p = 1$, i.e., for the group SO(10) $\otimes$ SO(10) with $p = 1$ copy of the $(\mathcal{S}, \mathcal{S})$ fermion. The structure of this theory is thus symmetric under interchange of the two factor groups. If we break this symmetry by setting one α_i to be large and the other small in Eq. (7.4), then we can obtain situations in which one SO(10) coupling dominates over the other. However, because of the structural symmetry, in contrast to the generic behavior that we have found for the other direct-product chiral gauge theories that we have investigated, here the pattern of symmetry breaking is the same regardless of which SO(10) gauge coupling is large and dominant.

If the first SO(10) gauge coupling gets large enough and dominates over the second SO(10) gauge coupling, or vice versa, this can produce fermion condensation in the channel

$$(\mathcal{S}, \mathcal{S}) \times (\mathcal{S}, \mathcal{S}) \to (\square_s, \square_s), \quad i.e.,$$

$$(16, 16) \times (16, 16) \to (10_s, 10_s) \ , \tag{7.100}$$

where we have used the fact that k and k' are even to infer the symmetry properties of $(\square, \square)$ in the Clebsch-Gordan decomposition of the direct product of the spinors. This condensation breaks the gauge symmetry SO(10) $\otimes$ SO(10) to SO(9) $\otimes$ SO(9), which is vectorial and does not break further.

7.8 SU(N) $\otimes$ SU(M) Theory

7.8.1 General Formulation

In this section we analyze a chiral gauge theory with a gauge group

$$G = \mathrm{SU}(N) \otimes \mathrm{SU}(M) \tag{7.101}$$

and fermion content consisting of a sum over $\mathcal{R}_{\mathrm{SU}(N)}$ and $\mathcal{R}_{\mathrm{SU}(M)}$ of

$$\begin{aligned} &\dim(\mathcal{R}_{\mathrm{SU}(M)})\,(\mathcal{R}_{\mathrm{SU}(N)},1) + (\bar{\mathcal{R}}_{\mathrm{SU}(N)},\bar{\mathcal{R}}_{\mathrm{SU}(M)}) \\ + \quad &\dim(\mathcal{R}_{\mathrm{SU}(N)})\,(1,\mathcal{R}_{\mathrm{SU}(M)}) \ , \end{aligned} \tag{7.102}$$

where $\mathcal{R}_{\mathrm{SU}(N)}$ and $\mathcal{R}_{\mathrm{SU}(M)}$ denote representations of SU(N) and SU(M), respectively. This theory is of type (cav, cav) in the classification of Section 7.1. A special case of this theory with $\mathcal{R}_{\mathrm{SU}(N)}$ and $\mathcal{R}_{\mathrm{SU}(M)}$ both equal to the fundamental representation was studied before in [83, 108], but in both of these previous works, it was studied as an example of a preon theory that might confine without spontaneous symmetry breaking and hence produce massless composite fermions. Here we consider it in a different way, as a theory that can self-break with bilinear fermion condensate formation, and we study the generalized theory with fermion representations higher than the fundamental.

The numbers $M \geq 2$ and $N \geq 2$, subject to the asymptotic freedom constraint (7.109) below. This is an irreducibly chiral gauge theory, so the chiral gauge invariance precludes any mass terms in the fundamental Lagrangian of the theory. One easily checks that this theory is free of any anomalies in gauged currents. It is also free of any global anomalies in the case where N or M is equal to 2. To see this, consider, for example, the case where $N = 2$ and the fermions that are nonsinglets under this group transform as doublets. From Eq. (7.102) one sees that the number of SU(2) doublets is $2\dim(\mathcal{R}_{\mathrm{SU}(M)})$ and hence is even.

We calculate the one-loop coefficients in the SU(N) and SU(M) beta functions to be

$$b_{1,\mathrm{SU}(N)} = \frac{1}{3}\Big[11N - 4\dim(\mathcal{R}_{\mathrm{SU}(M)})\,T(\mathcal{R}_{\mathrm{SU}(N)})\Big] \tag{7.103}$$

and

$$b_{1,\mathrm{SU}(M)} = \frac{1}{3}\Big[11M - 4\dim(\mathcal{R}_{\mathrm{SU}(N)})\,T(\mathcal{R}_{\mathrm{SU}(M)})\Big] \ . \tag{7.104}$$

Hence, the requirements that the SU(N) and SU(M) gauge interactions should be asymptotically free imply, respectively, that

$$\dim(\mathcal{R}_{\mathrm{SU}(M)})\,T(\mathcal{R}_{\mathrm{SU}(N)}) < \frac{11N}{4} \tag{7.105}$$

and

$$\dim(\mathcal{R}_{\mathrm{SU}(N)})\,T(\mathcal{R}_{\mathrm{SU}(M)}) < \frac{11M}{4} \ . \tag{7.106}$$

7.8.2 Model with Fermions (F, F)

Here we consider the version of the general theory of type (7.101) containing fermions with $\mathcal{R}_{\mathrm{SU}(N)} = \square$ and $\mathcal{R}_{\mathrm{SU}(M)} = \square$ (an equivalent notation is $F = \square$). Then

$$b_{1,\mathrm{SU}(N)} = \frac{1}{3}(11N - 2M) \tag{7.107}$$

and

$$b_{1,\mathrm{SU}(M)} = \frac{1}{3}(11M - 2N) \ , \tag{7.108}$$

so the inequalities (7.105) and (7.106) read $M < 11N/2$ and $N < 11M/2$, and the range of N and M allowed by these two constraints is given by

$$\frac{2}{11} < \frac{M}{N} < \frac{11}{2} \ . \tag{7.109}$$

We denote the fermion fields as

$$\omega^a_{i,L} \ , \quad i = 1, ..., M \ : \quad M\,(N, 1) \ , \tag{7.110}$$

$$\zeta_{a,\alpha,L} \ : \quad (\bar{N}, \bar{M}) \ , \tag{7.111}$$

and

$$\eta^\alpha_{j,L} \ , \quad j = 1, ..., N \ : \quad N\,(1, M) \ , \tag{7.112}$$

where a and α denote, respectively, SU(N) and SU(M) gauge indices and $i \in \{1, ..., M\}$ and and $j \in \{1, ..., N\}$ are copy (flavor) indices.

As noted, one possibility is confinement without any spontaneous chiral symmetry breaking, leading to massless composite spin 1/2 fermions that are singlets under SU(N) $\otimes$ SU(M). We investigate here the alternative possibility of condensate formation and associated chiral symmetry breaking. If the SU(N) gauge interaction is sufficiently strong and dominates over the SU(M) interaction, then this SU(N) interaction can produce condensation in the most attractive channel $N \times \bar{N} \to 1$. For the full theory, this is the channel

$$(N, 1) \times (\bar{N}, \bar{M}) \to (1, \bar{M}) \ , \tag{7.113}$$

with attractiveness measure given by $\Delta C_2 = 2C_2(N) = (N^2 - 1)/N$. The associated condensates are of the form

$$\langle \omega^{a\ T}_{i,L} C \zeta_{a,\alpha,L} \rangle \ , i = 1, ..., M \tag{7.114}$$

(where the sum over a here and below is from $a = 1$ to $a = N$). Consider the condensate (7.114) with $i = 1$. Since this transforms as a $\bar{M}$ representation of SU(M), it breaks this symmetry to SU($M-1$). By convention, we may use the initial SU(M) invariance to pick $\alpha = M$ in this condensate, so that it is

$$\langle \omega^{a\;T}_{1,L} C \zeta_{a,M,L} \rangle \; . \tag{7.115}$$

We denote the scale where this condensate forms as Λ. The fermions $\omega^a_{1,L}$ and $\zeta_{a,M,L}$ with $1 \le a \le N$ involved in this condensate thus gain dynamical masses of order Λ, as do the $2M-1$ gauge bosons in the coset SU(M)/SU($M-1$). In the resultant SU(N)$\otimes$SU($M-1$) chiral gauge theory, we consider the condensate (7.114) with $i = 2$ and $\alpha \in \{1, ..., M-1\}$. Again, by convention, we may use the residual SU($M-1$) gauge invariance to pick $\alpha = M-1$ in this condensate, so that it is

$$\langle \omega^{a\;T}_{2,L} C \zeta_{a,M-1,L} \rangle \; . \tag{7.116}$$

This preserves SU(N) and transforms like the conjugate fundamental representation of SU($M-1$), thereby breaking SU($M-1$) to SU($M-2$). This fermion condensation process continues with the formation of the condensates

$$\langle \omega^{a\;T}_{i,L} C \zeta_{a,M-i+1,L} \rangle \; , \quad i \le M \; , \tag{7.117}$$

breaking SU(M) completely. The last-enumerated condensate is $\langle \omega^{a\;T}_{M,L} C \zeta_{a,1,L} \rangle$. Since all of the condensates of the form (7.114) have the same attractiveness measure, ΔC_2, they are expected to form at approximately the same scale, Λ. All of the chiral fermions $\omega^a_{i,L}$ and $\zeta_{a,\alpha}$ with $1 \le i \le M$, $1 \le a \le N$, and $1 \le \alpha \le M$ are involved in these condensates and gain dynamical masses of order Λ, as do the full set of $M^2 - 1$ SU(M) gauge bosons. This leaves a theory with an SU(N) gauge invariance containing the $N^2 - 1$ SU(N) gauge bosons and a set of MN massless SU(N)-singlet fermions, namely the $\eta^\alpha_{j,L}$ with $1 \le \alpha \le M$ and $1 \le j \le N$. The SU(N) pure gluonic theory then forms a spectrum of SU(N)-singlet glueballs.

Clearly, if the SU(M) gauge interaction is sufficiently strong and dominates over the SU(N) gauge interaction, then the above discussion applies with the replacements $M \leftrightarrow N$ and $\omega^a_{i,L} \to \eta^\alpha_{j,L}$. In this case, the SU($M$) interaction breaks the SU(N) gauge symmetry completely, leaving the MN massless SU(M)-singlet fermions $\omega^a_{i,L}$ with $1 \le a \le N$ and $1 \le i \le M$. The SU(M) pure gluonic theory then forms a spectrum of SU(M)-singlet glueballs.

The version of the general theory with gauge group (7.101) and fermion representations $\mathcal{R}_{\mathrm{SU}(N)} = \Box$ and $\mathcal{R}_{\mathrm{SU}(M)} = \bar{\Box}$ exhibits the same properties as those that we have analyzed, with obvious changes, so we do not discuss it separately.

7.8.3 Model with (F, A_2)

We next analyze a model with the gauge group (7.101) and fermion representations $\mathcal{R}_{\mathrm{SU}(N)} = \Box$ and $\mathcal{R}_{\mathrm{SU}(M)} = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$. Since $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} = \bar{\Box}$ for $\mathrm{SU}(M) = \mathrm{SU}(3)$, we restrict $M \ge 4$. For this model the general equations (7.103) and (7.104) read

$$b_{1,\mathrm{SU}(N)} = \frac{1}{3}[11N - M(M-1)] \tag{7.118}$$

and

$$b_{1,\mathrm{SU}(M)} = \frac{1}{3}[11M - 2N(M-2)] \ . \tag{7.119}$$

The general inequalities (7.105) and (7.106) guaranteeing the asymptotic freedom of the $\mathrm{SU}(N)$ and $\mathrm{SU}(M)$ gauge interactions read, respectively,

$$N > \frac{M(M-1)}{11} \tag{7.120}$$

and

$$N < \frac{11M}{2(M-2)} \ . \tag{7.121}$$

The lower bound on N from (7.120) is $N > 2$ for $M = 4$ and increases as M increases. The upper bound on N from (7.121) is $N < 11$ for $M = 4$ and decreases as M increases. The curves for the upper and lower bounds on N as a function of M cross each other at

$$M = \frac{3 + 9\sqrt{3}}{2} = 9.294 \tag{7.122}$$

where

$$N = \frac{123 + 18\sqrt{3}}{22} = 7.008 \ , \tag{7.123}$$

The allowed values of M and N thus lie within the enclosed region between the upper and lower curves. This region has finite area and hence there are only finitely many allowed values of M and N. This is in contrast to the joint asymptotic freedom constraint for the model with (F, F) fermions, (7.109), which is an infinite wedge-shaped region in the M, N plane. As is evident, for a given $M \geq 4$, the range of allowed values of N decreases with increasing M. For $M = 4$, N may take on values in the range $2 \leq N \leq 10$, while for $M = 8$, the allowed values of N are $N = 6,\ 7$, and for $M = 9$, there is only one allowed value of N, namely $N = 7$. If $M \geq 10$, there are no values of N that satisfy the inequalities (7.120) and (7.121).

Chapter 8

Radiative Decays of Heavy-Quark Hadrons

In the paper [5], we extend our previous work with Ke and Li in Ref. [69] on the study of the radiative decays

$$h_c(1P) \to \eta_c(1S) + \gamma \ , \tag{8.1}$$

and

$$h_b(1P) \to \eta_b(1S) + \gamma \ . \tag{8.2}$$

We also carry out the reduction of the light-front amplitude to the non-relativistic limit, explicitly computing the leading and next-to-leading order relativistic corrections. This shows the consistency of the light-front approach with the non-relativistic formula for this electric dipole transition. Furthermore, we investigate the theoretical uncertainties in the predicted widths as functions of the inputs for the heavy quark mass and wavefunction structure parameters. As in Ref. [69], we compare our numerical results for these widths with experimental data and with other theoretical predictions from calculations based on non-relativistic models and their extensions to include relativistic effects, extending [69] with further study of the theoretical uncertainties in our calculations. Specifically, we compare our numerical results with results from [36,38,47–50,52,53] as well as latest experimental data [54]. This chapter will be a summary of results in the published paper [5].

8.1 Light-front formalism for the decays $1^{+-} \to 0^{-+} + \gamma$

8.1.1 Notation

We first define some notation, retaining the conventions of [63, 65]. In light-front coordinates, the four-momentum p is

$$p^\mu = (p^-, p^+, \mathbf{p}_\perp) \ , \tag{8.3}$$

where $p^\pm = p^0 \pm p^3$ and $\mathbf{p}_\perp = (p^1, p^2)$. Hence, the Lorentz scalar product $p^2 = p_\mu p^\mu$ is

$$p^2 = (p^0)^2 - |\mathbf{p}|^2 = (p^0)^2 - (p^3)^2 - |\mathbf{p}_\perp|^2 = p^+ p^- - |\mathbf{p}_\perp|^2 \ . \tag{8.4}$$

Consider a decay of a $Q\bar{Q}$ meson consisting of two constituent particles (quark and antiquark). The momentum of the parent meson is denoted as $P' = p'_1 + p_2$, where p'_1 and p_2 are the momenta of the constituent quark and antiquark, with mass m'_1 and m_2, respectively. The momentum of the daughter $Q\bar{Q}$ meson is written as $P'' = p''_1 + p_2$, where p''_1 is the momentum of the constituent quark, with mass m''_1. Here we have $m'_1 = m_2 = m'' = m_Q$. The four-momentum of the parent meson with mass M' can be expressed as $P' = (P'^-, P'^+, \mathbf{P}'_\perp)$, where $P'^2 = P'^+ P'^- - |\mathbf{P}'_\perp|^2 = M'^2$. Similarly, for the daughter meson with mass M'', one has $P''^2 = M''^2$, as shown in Fig. 8.1 below. (Vector signs on transverse momentum components are henceforth taken to be implicit.)

The momenta of the constituent quark and antiquark (p'_1, p''_1 and p_2) can be described by internal variables $(x_2, p'_\perp)$ thus:

$$\begin{aligned} &p'^+_1 = x_1 P'^+, \quad p^+_2 = x_2 P'^+ \\ &p'_{1\perp} = x_1 P'_\perp + p'_\perp, \quad p_{2\perp} = x_2 P'_\perp - p'_\perp \ , \quad x_1 + x_2 = 1 \ . \end{aligned} \tag{8.5}$$

Explicitly,

$$x_1 = \frac{e_1 - p'_z}{e'_1 + e_2} \ , \quad x_2 = \frac{p'_z + e_2}{e'_1 + e_2} \ , \tag{8.6}$$

where e'_1, e''_1 and e_2 are the energy of the quark (anti-quark) with momenta

p'_1, p''_1 and p_2:

$$\begin{aligned} e'_1 &= \sqrt{m_1'^2 + p_\perp'^2 + p_z'^2} \\ e''_1 &= \sqrt{m_1''^2 + p_\perp''^2 + p_z''^2} \\ e_2 &= \sqrt{m_2^2 + p_\perp'^2 + p_z'^2} \ . \end{aligned} \tag{8.7}$$

With the external momentum of the photon given as $q = P' - P''$, $p''_\perp$ can be expressed as

$$p''_\perp = p'_\perp - x_2 q_\perp. \tag{8.8}$$

Here p'_z and p''_z can also be expressed as functions of internal variables $(x_2, p'_\perp)$, and explicit expressions can be found in C.

8.1.2 Form factors

Define external momentum variables to be $P = P' + P''$, $q = P' - P''$, where q is the four-momentum of the photon that is emitted in the radiative transition. The general amplitude of the radiative decay (1.15) of the axial vector 1^{+-} $^1P_1$ meson, denoted as A, to the pseudoscalar 0^{-+} $^1S_0$ meson, denoted as P, can be written as [65]:

$$i\mathcal{A}\left(A(P') \to P(P'')\gamma(q)\right) = \varepsilon^*_\mu(q)\varepsilon'_\nu(P') i\tilde{\mathcal{A}}^{\mu\nu} \ , \tag{8.9}$$

where

$$i\tilde{\mathcal{A}}^{\mu\nu} = f_1(q^2)g^{\mu\nu} + P^\mu \left[f_+(q^2)P^\nu + f_-(q^2)q^\nu\right] \ . \tag{8.10}$$

In the above expression, we have used the condition $\varepsilon^*_\mu(q)q^\mu = 0$ to eliminate terms that are proportional to q^μ. This expression can be simplified further by using the transversality property of axial vector polarization vector:

$$\varepsilon'_\nu(P')(P+q)^\nu = 0 \ . \tag{8.11}$$

Then the general amplitude can be written as

$$i\tilde{\mathcal{A}}^{\mu\nu} \to i\mathcal{A}^{\mu\nu} = f_1(q^2)g^{\mu\nu} + f_2(q^2)P^\mu q^\nu \ , \tag{8.12}$$

where $f_2(q^2)$ is linear combination of $f_+(q^2)$ and $f_-(q^2)$:

$$f_2(q^2) = -f_+(q^2) + f_-(q^2) \ . \tag{8.13}$$

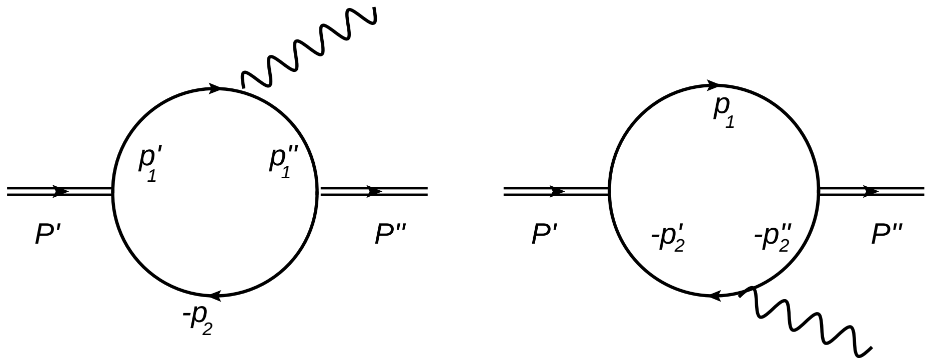

Figure 8.1: Feynman diagrams for radiative transition $1^{+-} \to 0^{-+} + \gamma$ in the light-front approach.

Notice that in Eq.(8.12), $f_1(q^2)$ and $f_2(q^2)$ are not independent. Because of electromagnetic gauge invariance, they are related by the following equation:

$$q_\mu \mathcal{A}^{\mu\nu} = 0 \;\to\; f_1(q^2) + f_2(q^2)(P \cdot q) = 0 \; . \tag{8.14}$$

So the amplitude can be parameterized by $f_1(q^2)$, which is

$$i\mathcal{A}^{\mu\nu} = f_1(q^2)\left[g^{\mu\nu} - \frac{1}{(P \cdot q)} P^\mu q^\nu\right] \; . \tag{8.15}$$

After an explicit calculation, we have

$$\sum_{\text{polarization}} |\mathcal{A}|^2 = 2|f_1(q^2)|^2 \; . \tag{8.16}$$

Taking the physical value $q^2 \to 0$ in the form factor $f_1(q^2)$ and averaging initial state polarizations, the radiative transition width of $1^{+-} \to 0^{-+} + \gamma$ is given by

$$\Gamma = \frac{1}{3} \cdot \frac{|\mathbf{q}|}{8\pi M'^2} \sum_{\text{polar.}} |\mathcal{A}|^2 = \frac{|\mathbf{q}|}{12\pi M'^2} \cdot |f_1(0)|^2 \; , \tag{8.17}$$

where the energy of the emitted photon is related to the masses of mesons as $|\mathbf{q}| = (M'^2 - M''^2)/(2M')$.

8.1.3 Calculation of radiative decay amplitude

In the covariant light-front quark model, the vertex function of the axial vector meson A $(1^{+-},{}^1P_1)$ is given by

$$-iH'_A\left[\frac{1}{W'_A}(p'_1-p_2)^\mu\right]\gamma^5 \ , \tag{8.18}$$

and the vertex function of the pseudoscalar meson P $(0^{-+},{}^1S_0)$ is given by

$$H''_P\gamma^5 \ , \tag{8.19}$$

where H'_A and H''_P are functions of p'_1 and p_2, and W'_A can be reduced to a constant, which we will discuss later in this subsection.

In the light-front framework that we use [63, 65], at leading order there are two diagrams that contribute to the $A \to P+\gamma$ transition amplitude. These give the corresponding contributions to this amplitude

$$i\mathcal{A}^{\mu\nu}(A\to P+\gamma) = i\mathcal{A}^{\mu\nu}(a) + i\mathcal{A}^{\mu\nu}(b) \ , \tag{8.20}$$

where $i\mathcal{A}^{\mu\nu}(a)$ and $i\mathcal{A}^{\mu\nu}(b)$ correspond to the left and right diagram in Fig.(8.1), respectively. The contribution to the amplitude from the right diagram can be obtained by taking the charge conjugation of left diagram (see also [70]). So we discuss the left-hand diagram, where the corresponding transition amplitude is given by

$$i\mathcal{A}^{\mu\nu}(a) = i\frac{eN_{e'_1}N_c}{(2\pi)^4}\int d^4p'_1\,\frac{H'_AH''_P}{N'_1N''_1N_2}\mathcal{S}^{\mu\nu}_a \ , \tag{8.21}$$

where

$$\begin{aligned}
\mathcal{S}^{\mu\nu}_a &= \mathrm{Tr}[\gamma^5(\not{p}''_1+\mathrm{m}''_1)\gamma^\mu(\not{p}'_1+\mathrm{m}'_1)\gamma^5(-\not{p}_2+\mathrm{m}_2)]\frac{1}{\mathrm{W'_A}}(2\mathrm{p}'_1-\frac{\mathrm{P+q}}{2})^\nu \\
&= \frac{4}{W'_A}(2p'_1-\frac{P+q}{2})^\nu[p''^\mu_1(p'_1\cdot p_2)+p'^\mu_1(p''_1\cdot p_2) \\
&- p^\mu_2(p'_1\cdot p''_1)+m'_1m_2p''^\mu_1+m''_1m_2p'^\mu_1+m'_1m''_1p^\mu_2] \\
&= \frac{1}{W'_A}(2p'_1-\frac{P+q}{2})^\nu\{2p'^\mu_1[M'^2+M''^2-q^2-2N_2 \\
&- (m'_1-m_2)^2-(m''_1-m_2)^2+(m'_1-m''_1)^2] \\
&+ q^\mu[q^2-2M'^2+N'_1-N''_1+2N_2+2(m'_1-m_2)^2 \\
&- (m'_1-m''_1)^2]+P^\mu[q^2-N'_1-N''_1-(m'_1-m''_1)^2]\} \ ,
\end{aligned} \tag{8.22}$$

$$
\begin{aligned}
&N_1' = p_1'^2 - m_1'^2 + i\epsilon, \quad N_1'' = p_1''^2 - m_1''^2 + i\epsilon,\\
&N_2 = p_2^2 - m_2^2 + i\epsilon.
\end{aligned}
\tag{8.23}
$$

Here $N_{e_1'(e_2)}$ represents the electric charge of quark with four momentum p_1' (p_2). Here we have $N_{e_1'(e_2)} = e_Q$. In Eq.(8.22), we have already applied the following relations:

$$
\begin{aligned}
p_1'' &= p_1' - q\\
p_2 &= (P+q)/2 - p_1'\\
2p_1' \cdot p_2 &= M'^2 - N_1' - m_1'^2 - N_2 - m_2^2\\
2p_1'' \cdot p_2 &= M''^2 - N_1'' - m_1''^2 - N_2 - m_2^2\\
2p_1' \cdot p_1'' &= -q^2 + N_1' + m_1'^2 + N_1'' + m_1''^2.
\end{aligned}
\tag{8.24}
$$

Then we integrate over $p_1'^-$ by closing the contour in the upper complex $p_1'^-$ plane, which amounts to the following replacement [63, 65]:

$$
\int d^4p_1' \frac{H_A' H_P''}{N_1' N_1'' N_2} \mathcal{S}_a^{\mu\nu} \varepsilon_\mu^* \varepsilon_\nu' \to -i\pi \int dx_2 d^2 p_\perp' \frac{h_A' h_P''}{x_2 \hat{N}_1' \hat{N}_1''} \hat{\mathcal{S}}_a^{\mu\nu} \hat{\varepsilon}_\mu^* \hat{\varepsilon}_\nu' \ , \tag{8.25}
$$

where

$$
\begin{aligned}
&N_1' \to \hat{N}_1' = x_1(M'^2 - M_0'^2)\\
&N_1'' \to \hat{N}_1'' = x_1(M''^2 - M_0''^2)\\
&H_A' \to h_A' = (M'^2 - M_0'^2)\sqrt{\frac{x_1 x_2}{N_c}} \frac{1}{\sqrt{2}\tilde{M}_0'} \varphi_p(p_\perp', x_2)\\
&H_P'' \to h_P'' = (M''^2 - M_0''^2)\sqrt{\frac{x_1 x_2}{N_c}} \frac{1}{\sqrt{2}\tilde{M}_0''} \varphi(p_\perp'', x_2)\\
&W_A' \to w_A' = 2\ .
\end{aligned}
\tag{8.26}
$$

In the above expressions, $\varphi_p(p_\perp', x_2)$ is the light-front momentum space wavefunction for initial P-wave meson ($^1P_1$), and $\varphi(p_\perp'', x_2)$ is the wavefunction for the final S-wave meson, $^1S_0$. Some details concerning the wavefunctions are given in B. The explicit forms of M_0', M_0'', $\tilde{M}_0'$ and $\tilde{M}_0''$ are listed in C. The definitions of $\hat{\varepsilon}^*$, $\hat{\varepsilon}'$ and $\hat{\varepsilon}_\rho''^*$ are given in [63, 65].

After the integration over $p_1'^-$, we have the following replacement for $p_{1\mu}'$

and $\hat{N}_2$ in $\hat{\mathcal{S}}_a^{\mu\nu}$ in the integral [63, 65]:

$$
\begin{aligned}
\hat{p}'_{1\mu} &\to P_\mu A_1^{(1)} + q_\mu A_2^{(1)} \ , \\
\hat{p}'_{1\mu}\hat{p}'_{1\nu} &\to g_{\mu\nu} A_1^{(2)} + P_\mu P_\nu A_2^{(2)} \ , \\
&+ (P_\mu q_\nu + q_\mu P_\nu) A_3^{(2)} + q_\mu q_\nu A_4^{(2)} \ , \\
\hat{p}'_{1\mu}\hat{N}_2 &\to q_\mu [A_2^{(1)} Z_2 + \frac{q\cdot P}{q^2} A_1^{(2)}] \ , \\
\hat{p}'_{1\mu}\hat{p}'_{1\nu}\hat{N}_2 &\to g_{\mu\nu} A_1^{(2)} Z_2 + q_\mu q_\nu [A_4^{(2)} Z_2 + 2\frac{q\cdot P}{q^2} A_2^{(1)} A_1^{(2)}],
\end{aligned}
\tag{8.27}
$$

where the explicit expressions for $A_j^{(i)}(i, j = 1 \sim 4)$ and Z_2 are listed in C.

Combining Eq.(8.25), Eq.(8.26) and Eq.(8.27), we get $\mathcal{S}_a^{\mu\nu} \to \hat{\mathcal{S}}_a^{\mu\nu}$, where the explicit form can be found in Ref. [69] . Finally, we obtain $i\mathcal{A}^{\mu\nu}(a)$ as a function of the external four-momenta P and q with the following parameterization:

$$
i\mathcal{A}^{\mu\nu}(a) = f_1^a(q^2)\left[g^{\mu\nu} - \frac{1}{(P\cdot q)} P^\mu q^\nu\right] \ , \tag{8.28}
$$

where the form factor $f_1^a(q^2)$ is given by

$$
\begin{aligned}
f_1^a(q^2) &= \frac{ee_Q N_c}{16\pi^3} \int \frac{dx_2 d^2 p'_\perp}{x_2 \hat{N}'_1 \hat{N}''_1} h'_A h''_P \frac{4}{w'_A} \{A_1^{(2)}[M'^2 + M''^2 \\
&- q^2 - (m'_1 - m_2)^2 - (m''_1 - m_2)^2 + (m'_1 - m''_1)^2] - 2A_1^{(2)} Z_2\} \\
&= \frac{ee_Q N_c}{16\pi^3} \int \frac{dx_2 d^2 p'_\perp}{x_2 \hat{N}'_1 \hat{N}''_1} h'_A h''_P \frac{4}{w'_A} (-p'^2_\perp - \frac{(p'_\perp\cdot q_\perp)^2}{q^2}) \{[M'^2 + M''^2 \\
&- q^2 - (m'_1 - m_2)^2 - (m''_1 - m_2)^2 \\
&+ (m'_1 - m''_1)^2] - 2\hat{N}'_1 - 2m'^2_1 + 2m_2^2 \\
&- 2(1-2x_1)M'^2 - 2(q^2 + q\cdot P)\frac{p'_\perp\cdot q_\perp}{q^2}\} \\
&= \frac{ee_Q}{16\pi^3} \int \frac{dx_2 d^2 p'_\perp}{x_1 M'_0 M''_0} \varphi_p(p'_\perp, x_2)\varphi(p''_\perp, x_2) \left[-p'^2_\perp - \frac{(p'_\perp\cdot q_\perp)^2}{q^2}\right] \\
&\times [(2x_1 - 1)M'^2 + M''^2 + 2x_1 M'^2_0 - q^2 - 2(q^2 + q\cdot P)\frac{p'_\perp\cdot q_\perp}{q^2}].
\end{aligned}
\tag{8.29}
$$

Similarly, for the right diagram in Fig.(8.1), we have the corresponding amplitude:

$$i\mathcal{A}^{\mu\nu}(b) = f_1^b(q^2)\left[g^{\mu\nu} - \frac{1}{(P\cdot q)}P^\mu q^\nu\right] \ . \tag{8.30}$$

This can be obtained from the result of the left-hand diagram with the replacements $m_1' \leftrightarrow m_2'$, $m_1'' \leftrightarrow m_2''$, $m_2 \leftrightarrow m_1$, $N_{e_1'} \leftrightarrow N_{e_2}$. The total form factor $f_1(q^2)$ is the sum of contribution from two diagrams:

$$f_1(q^2) = f_1^a(q^2) + f_1^b(q^2) \ . \tag{8.31}$$

Taking the physical value $q^2 \to 0$ in the form factor, we can obtain $|f_1(0)|^2$ and compute the decay width using Eq.(8.17). The numerical calculation of the decay width will be discussed in Section 8.3.

8.1.4 Comments on effects of zero modes

As discussed in Refs. [63] and [65], there are two classes of form factors for the amplitude discussed in this section. One class of form factors like $f(q^2)$ is associated with zero modes and another class of form factors is free of zero-mode contributions. In this analysis, the form factor that contributes to the radiative transition $1^{+-} \to 0^{-+} + \gamma$, namely $f_1(q^2)$, belongs to the first class and contains zero-mode contributions. In this case, the substitution in Eq. (8.27) is not exact and contains residual spurious terms that are proportional to a lightlike four-vector $\omega = (2, 0, 0)$. These terms are not Lorentz-covariant. As discussed in Refs. [63,65], the zero-mode contribution cancels away the residual spurious ω terms. Furthermore, form factors like $f(q^2)$ receive additional residual contributions, which can be expressed in terms of the $B_{(n)}^{(m)}$ and $C_{(n)}^{(m)}$ functions defined in Appendix B of Ref. [65].

However, this problem has already been carefully analyzed in Ref. [65]. In fact, by comparing our expression for $f_1(q^2)$ in Eq.(8.29) with the expression for $f(q^2)$ in Eq. (B4) of Ref. [65], we find that the integrand of $f_1(q^2)$ is proportional to the $1/w_V''$ term of the integrand of $f(q^2)$ in Ref. [65]. Thus, we follow the same analysis in Ref. [65] of $f(q^2)$ to address the possible zero-mode problem of $f_1(q^2)$. The $B_{(n)}^{(m)}$ and $C_{(n)}^{(m)}$ functions do not appear in the expression for $f_1(q^2)$, so we can still use the substitution in Eq. (8.27) because, first, we utilize the relation $C_1^{(1)} \to 0$ in the integrand, and second, the amplitude associated with $f_1(q^2)$ in $\mathcal{A}^{\mu\nu}$ is contracted with the transverse polarization vector of the photon. Explicitly, our numerical calculation shows that all of the functions $B_{(n)}^{(m)}$ are numerically negligibly small, and hence we can neglect all of the residual contributions to the form factors in the present analysis.

8.2 Reduction to non-relativistic limit in application to quarkonium systems

In this chapter, we use the light-front formula discussed in Section 8.1 to study the radiative decay (1.15). For this decay, the non-relativistic electromagnetic dipole transition formulas are widely adopted [36]. Thus it is interesting to investigate the consistency between the LFQM and the non-relativistic dipole transition formulas in the non-relativistic limit. In this section we analyze the reduction of the light-front formula for the decay width in the non-relativistic limit. This limit is relevant here because $(v/c)^2$ is substantially smaller than unity for a heavy-quark $Q\bar{Q}$ state. For a Coulombic potential, $\alpha_s \sim v/c$, and current data give $\alpha_s = 0.21$ at a scale of $m_b = 4.5$ GeV, yielding $(v/c)^2 \sim 0.04$ for the Υ system. There are several aspects of the non-relativistic limit for the decay of a heavy quarkonium system:

1. Masses of bound states.

   The masses of initial (M') and final state (M'') are close to the sum their constituents, and the deviation is $\mathcal{O}(m_Q^{-2})$ corrections:

   $$\frac{M'^2}{4m_Q^2} = 1 + \mathcal{O}(m_Q^{-2}) \ , \ \frac{M''^2}{4m_Q^2} = 1 + \mathcal{O}(m_Q^{-2}) \ . \tag{8.32}$$

   Here and below, by $\mathcal{O}(m_Q^{-2})$ we mean $\mathcal{O}(|\mathbf{p}|^2/m_Q^2)$, where $\mathbf{p}$ is a generic three-momentum in the parent meson rest frame.

2. No-recoil limit.

   In non-relativistic quantum mechanics, the final state after the E1 radiative transition is assumed to carry approximately the same three-momentum as the initial state [119]. So the matrix element of this E1 transition is

   $$\langle \mathbf{r} \rangle \propto \langle f(\mathbf{p}'')|\mathbf{r}|i(\mathbf{p}')\rangle, \quad \mathbf{p}'' = \mathbf{p}' \ . \tag{8.33}$$

   In our analysis, we will adopt this no-recoil approximation.

3. Normalization of wavefunction.

   In non-relativistic quantum mechanics, the momentum-space wavefunction is given by

$$\langle \mathbf{p}|n, lm\rangle = R_{nl}(p) Y_{lm}(\theta, \phi) \ , \tag{8.34}$$

   with the normalization of the radial wavefunction

$$\int_0^\infty dp\, p^2 R^*_{nl}(p) R_{nl}(p) = 1 \ , \tag{8.35}$$

   where $p = |\mathbf{p}|$, and the normalization of the angular wavefunction

$$\int d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'} \ . \tag{8.36}$$

In this chapter we use harmonic oscillator wavefunctions for the quarkonium 1P and 1S states. The general formula for harmonic oscillator wavefunctions in momentum space that satisfy the usual quantum mechanics normalization in Eq.(8.35) is given by [49, 120]

$$R_{nl}(p) = \frac{1}{\beta^{\frac{3}{2}}}\sqrt{\frac{2n}{\Gamma(n+l+\frac{1}{2})}}(\frac{p}{\beta})^l L^{l+\frac{1}{2}}_{n-1}(\frac{p^2}{\beta^2}) \exp(-\frac{p^2}{2\beta^2}) \ , \tag{8.37}$$

where $L^{l+\frac{1}{2}}_{n-1}(p^2/\beta^2)$ is an associated Laguerre polynomial. Here, β is a parameter with dimensions of momentum that enters in the light-front wavefunction (B.7) (and should not be confused with the dimensionless ratio v/c that serves as a measure of the non-relativistic property of a heavy-quark $Q\bar{Q}$ bound state.) Specifically, for 1S and 1P states, we have

$$R_{1\mathrm{S}}(p) = \frac{2}{\beta^{\frac{3}{2}}\pi^{\frac{1}{4}}} \exp(-\frac{p^2}{2\beta^2}) \ , \tag{8.38}$$

and

$$R_{1\mathrm{P}}(p) = \sqrt{\frac{2}{3}}\frac{2}{\beta^{\frac{3}{2}}\pi^{\frac{1}{4}}} \exp(-\frac{p^2}{2\beta^2})\frac{p}{\beta} \ . \tag{8.39}$$

Notice that the normalization of these wavefunctions is different from the normalization of the light-front wavefunctions discussed in B. For example,

$$\psi(p) = \frac{1}{\sqrt{4\pi}} R_{1\mathrm{S}}(p) \ . \tag{8.40}$$

In non-relativistic quantum mechanics, the width of an E1 decay of the initial quarkonium state ${}^1P_1$ to the final quarkonium state ${}^1S_0 + \gamma$ is given by [36]:

$$\Gamma({}^1P_1 \to {}^1S_0 + \gamma) = \frac{4}{9}\alpha e_Q^2 E_\gamma^3 |I_3(1\mathrm{P} \to 1\mathrm{S})|^2 \ , \tag{8.41}$$

where $E_\gamma = |\mathbf{q}|$ is the energy of the emitted photon, and $I_3(1\mathrm{P} \to 1\mathrm{S})$ is the overlap integral in position space, which represents the matrix element of the electric dipole operator:

$$I_3(1\mathrm{P} \to 1\mathrm{S}) = \int_0^\infty dr\, r^3 R_{1\mathrm{P}}(r) R^*_{1\mathrm{S}}(r) \ . \tag{8.42}$$

Similarly, we can define $I_5(1\mathrm{P} \to 1\mathrm{S})$, which appears in the relativistic correction to the electric dipole transition width [36]:

$$I_5(1\mathrm{P} \to 1\mathrm{S}) = \int_0^\infty dr\, r^5 R_{1\mathrm{P}}(r) R^*_{1\mathrm{S}}(r) \ . \tag{8.43}$$

For later use, we also list the analogous integrals in momentum space:

$$I_3^p(1\mathrm{P} \to 1\mathrm{S}) = \int_0^\infty dp\, p^3 R_{1\mathrm{P}}(p) R^*_{1\mathrm{S}}(p) \ , \tag{8.44}$$

$$I_5^p(1\mathrm{P} \to 1\mathrm{S}) = \int_0^\infty dp\, p^5 R_{1\mathrm{P}}(p) R^*_{1\mathrm{S}}(p) \ . \tag{8.45}$$

We are now ready to reduce the light-front decay width in Eq.(8.17) when applied to quarkonium systems to the standard non-relativistic formula in Eq. (8.41). Using the explicit form in Eq.(8.29) and taking the limit $q^2 \to 0$, the form factor in Eq.(8.31), we can write:

$$\begin{aligned}
f_1(q^2) &= \frac{2ee_Q}{16\pi^3} \int \frac{dx_2 d^2p'_\perp}{x_1 M'_0 M''_0} \varphi_p(p'_\perp, x_2)\varphi(p''_\perp, x_2) \left[-p'^2_\perp - \frac{(p'_\perp \cdot q_\perp)^2}{q^2} \right] \\
&\times [(2x_1 - 1)M'^2 + M''^2 + 2x_1 M'^2_0 - 2(q \cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}] \\
&= -ee_Q \int \frac{dx_2 d^2p'_\perp}{x_1 M'_0 M''_0} \sqrt{\frac{dp'_z}{dx_2}} \sqrt{\frac{dp'_z}{dx_2}} \sqrt{\frac{e''_1 M'_0}{e'_1 M''_0}} \psi_p(p'_\perp, p'_z)\psi(p''_\perp, p''_z) p'^2_\perp \\
&\times [(2x_1 - 1)M'^2 + M''^2 + 2x_1 M'^2_0 - 2(q \cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}] \ ,
\end{aligned} \tag{8.46}$$

where we use the explicit form of light-front momentum space wavefunction in B. This expression can be further simplified in the no-recoil limit, which is a valid approximation in the study of an electric dipole transition in the non-relativistic limit [119]. In this limit, we have $\sqrt{\frac{e_1'' M_0'}{e_1' M_0''}} \to 1$, $M_0'' \to M_0'$ and $\psi(p_\perp''^2, p_z''^2) \to \psi(p_\perp'^2, p_z'^2)$. The corrections due to recoil effect are suppressed by powers of $(1/m_Q)$:

$$\sqrt{\frac{e_1'' M_0'}{e_1' M_0''}} = \sqrt{\frac{2\sqrt{\mathbf{p}''^2 + m_Q^2}}{\sqrt{\mathbf{p}'^2 + m_Q^2} + \sqrt{\mathbf{p}''^2 + m_Q^2}}} = 1 - \frac{1}{8}\frac{(\mathbf{p}'^2 - \mathbf{p}''^2)^2}{m_Q^4} + \mathcal{O}(m_Q^{-6}) \ , \tag{8.47}$$

$$M_0'' = M_0' + \frac{1}{2}\frac{(\mathbf{p}''^2 - \mathbf{p}'^2)}{m_Q} + \mathcal{O}(m_Q^{-3}) \ . \tag{8.48}$$

The last term in Eq.(8.46), $-(q \cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}$, requires a more careful treatment. It seems that linear $p'_\perp$ terms will not make contributions after integrating over $p'_\perp$, but the Taylor expansion of the functions of $p''_\perp$ in the integrands will generate a term that is proportional to $(p'_\perp \cdot q_\perp)$, and this can combine with $-(q \cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}$ term to produce a q^2 independent term, which is non-zero in the physical $q^2 \to 0$ and no-recoil limit. Firstly we should expand $p''_\perp$ in powers of inverse of m_Q:

$$\begin{aligned} p''_\perp &= p'_\perp - x_2 q_\perp = p'_\perp - q_\perp \left(\frac{1}{2} + \frac{p'_z}{2\sqrt{m_Q^2 + p'^2_\perp + p'^2_z}} \right) \\ &= p'_\perp - \frac{1}{2}q_\perp - \frac{1}{2}\frac{p'_z}{m_Q}q_\perp + \mathcal{O}(m_Q^{-2}) \ . \end{aligned} \tag{8.49}$$

We find in the physical limit $q^2 \to 0$, the dominant contribution to the $(p'_\perp \cdot q_\perp)$ term comes from the expansion of $\psi(p''_\perp, p''_z)$. Since $\psi(p''_\perp, p''_z)$ is the wavefunction of the 1S state, it is a function of $\mathbf{p}''^2$. Hence, we can write $\psi(p''_\perp, p''_z) = \psi(p''^2_\perp, p''^2_z)$ and expand it as follows:

$$\begin{aligned} \psi(p''^2_\perp, p''^2_z) &\approx \psi(p'^2_\perp, p'^2_z) - p'_\perp \cdot q_\perp [\frac{d\psi}{dp'^2_\perp}] + \mathcal{O}(m_Q^{-2}) \\ &= \psi(p'^2_\perp, p'^2_z) + p'_\perp \cdot q_\perp \frac{1}{2\beta^2}\psi(p'^2_\perp, p'^2_z) + \mathcal{O}(m_Q^{-2}) \ , \end{aligned} \tag{8.50}$$

where we use the explicit form of $\psi(p''_\perp, p''_z) \propto \exp[-\mathbf{p}''^2/(2\beta^2)]$ to calculate its derivative. Plugging the expansion of $\psi(p''_\perp, p''_z)$ into the integrands, we find the contribution of the $-(q\cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}$ terms is

$$\begin{aligned}
\int ...\psi(p''_\perp, p''_z)[-(q\cdot P)\frac{p'_\perp \cdot q_\perp}{q^2}]|_{q^2\to 0} &= -(q\cdot P)\int ...[\frac{d\psi}{dp'^2_\perp}][\frac{(p'_\perp \cdot q_\perp)^2}{q^2_\perp}] \\
&= \frac{1}{2\beta^2}(q\cdot P)\int ...\psi(p'_\perp, p'_z)\frac{1}{2}p'^2_\perp \,. \qquad (8.51)
\end{aligned}$$

After this calculation, in the physical $q^2 = 0$ and no-recoil limit, the form factor $f_1(q^2 \to 0)$ is given by

$$\begin{aligned}
f_1(0) &\approx -ee_Q \int \frac{dp'_z d^2p'_\perp}{x_1 M'^2_0}\psi_p(p'_\perp, p'_z)\psi(p'_\perp, p'_z)p'^2_\perp \cdot [(2x_1 - 1)M'^2 + M''^2 \\
&+ 2x_1 M'^2_0 + 2(q\cdot P)\frac{1}{4\beta^2}p'^2_\perp] \\
&= -ee_Q \int dp'_z d^2p'_\perp \psi_p(p'_\perp, p'_z)\psi(p'_\perp, p'_z)p'^2_\perp \cdot \left[2 + \frac{2M'^2}{4(m_Q^2 + p'^2_\perp + p'^2_z)}\right. \\
&+ \frac{M''^2 - M'^2}{2(m_Q^2 + p'^2_\perp + p'^2_z) - 2p'_z\sqrt{m_Q^2 + p'^2_\perp + p'^2_z}} \\
&+ \left.\frac{2|\mathbf{q}|M'}{(m_Q^2 + p'^2_\perp + p'^2_z) - p'_z\sqrt{m_Q^2 + p'^2_\perp + p'^2_z}}\frac{1}{4\beta^2}p'^2_\perp\right], \qquad (8.52)
\end{aligned}$$

where we use the kinematic relation $(q\cdot P) = 2|\mathbf{q}|M'$. In the non-relativistic limit, it is more convenient to use notation of wavefunctions in non-relativistic quantum mechanics. Using Eq. (8.40), $f_1(q^2 \to 0)$ can be rewritten as

$$\begin{aligned}
f_1(q^2 \to 0) &\approx -\frac{1}{4\pi}\cdot\frac{\sqrt{2}}{\beta}\cdot ee_Q \int d^3\mathbf{p}'\, R_{1S}(\mathbf{p}')R_{1S}(\mathbf{p}')p'^2_\perp \\
&\times \left[2 + \frac{2M'^2}{4(m_Q^2 + \mathbf{p}'^2)} + \frac{M''^2 - M'^2}{2(m_Q^2 + \mathbf{p}'^2) - 2p'_z\sqrt{m_Q^2 + \mathbf{p}'^2}}\right. \\
&+ \left.\frac{2|\mathbf{q}|M'}{(m_Q^2 + \mathbf{p}'^2) - p'_z\sqrt{m_Q^2 + \mathbf{p}'^2}}\frac{1}{4\beta^2}p'^2_\perp\right] \,. \qquad (8.53)
\end{aligned}$$

8.2.1 Leading-order non-relativistic approximation

In the leading-order non-relativistic approximation, we neglect the $\mathcal{O}(m_Q^{-2})$ contributions in Eq.(8.53), so $f_1(q^2 \to 0)$ is given by

$$\begin{aligned} f_1(q^2 \to 0) &\approx -4\pi \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int d^3\mathbf{p}'\, R_{1S}(\mathbf{p}') R_{1S}(\mathbf{p}') p_\perp'^2 \\ &\times \left[2 + \frac{2M'^2}{4m_Q^2} + \frac{M''^2 - M'^2}{2m_Q^2} + \frac{2|\mathbf{q}|M'}{m_Q^2} \frac{1}{4\beta^2} p_\perp'^2 + \mathcal{O}(m_Q^{-4}) \right] \\ &= -\frac{1}{4\pi} \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int d^3\mathbf{p}'\, R_{1S}(\mathbf{p}') R_{1S}(\mathbf{p}') p_\perp'^2 \cdot 4 + \mathcal{O}(m_Q^{-2}) \;. \end{aligned} \tag{8.54}$$

This integral can be simplified by using symmetric property of functions in the integrands. For function $F(\mathbf{p}^2)$ that have spherical symmetry, the following relation is satisfied:

$$\int d^3\mathbf{p}\, F(\mathbf{p}^2) p_i p_j = \frac{1}{3}\delta_{ij} \int d^3\mathbf{p}\, F(\mathbf{p}^2)\mathbf{p}^2 \;. \tag{8.55}$$

So Eq.(8.54) can be written as

$$\begin{aligned} f_1(0) &= -\frac{4}{4\pi} \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int d^3\mathbf{p}'\, R_{1S}(\mathbf{p}') R_{1S}(\mathbf{p}') p_\perp'^2 \\ &= -\frac{2}{3} \cdot \frac{4}{4\pi} \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int d^3\mathbf{p}'\, R_{1S}(\mathbf{p}') R_{1S}(\mathbf{p}') \mathbf{p}'^2 \\ &= -\frac{2}{3} \cdot 4 \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int_0^\infty dp'\, p'^4 R_{1S}(p') R_{1S}(p') \;, \end{aligned} \tag{8.56}$$

where p' denotes the radial coordinate in the three dimensional momentum space (and should not be confused with a four-momentum). Using the definition of wavefunction in Eq.(8.39),

$$R_{1P}(p') = \frac{1}{\beta}\sqrt{\frac{2}{3}} R_{1S}(p') p' \;, \tag{8.57}$$

we find that this integral is proportional to $I_3^p(1\mathrm{P} \to 1\mathrm{S})$:

$$\begin{aligned} f_1(0) &= -\frac{2}{3} \cdot 4 \cdot \frac{\sqrt{2}}{\beta} \cdot ee_Q \int_0^\infty dp'\, p'^4 R_{1\mathrm{S}}(p') R_{1\mathrm{S}}(p') \\ &= -\sqrt{\frac{3}{2}} \cdot \frac{2}{3} \cdot 4 \cdot \sqrt{2} \cdot ee_Q \int_0^\infty dp'\, p'^3 R_{1\mathrm{P}}(p') R_{1\mathrm{S}}(p') \\ &= -\sqrt{\frac{3}{2}} \cdot \frac{2}{3} \cdot 4 \cdot \sqrt{2} \cdot ee_Q \cdot I_3^p(1\mathrm{P} \to 1\mathrm{S}) \ . \end{aligned} \tag{8.58}$$

Now $I_3^p(1\mathrm{P} \to 1\mathrm{S})$ is proportional to $I_3(1\mathrm{P} \to 1\mathrm{S})$, which is evident in non-relativistic quantum mechanics, where we have the operator relation:

$$\frac{\mathbf{p}}{m} = i[H, \mathbf{r}] \ , \tag{8.59}$$

so that

$$|\langle f|\frac{\mathbf{p}}{m}|i\rangle| = |\langle f[H,\mathbf{r}]|i\rangle| = (E_i - E_f)|\langle f|\mathbf{r}|i\rangle| \ . \tag{8.60}$$

In the non-relativistic limit, the mass can be interpreted as the reduced mass of the $\bar{Q}Q$ two-body system $m = \mu' = m_Q/2$, and in non-relativistic quantum mechanics the photon energy is the difference of energy levels between initial and final state, $E_i - E_f \approx |\mathbf{q}|$, so we have

$$I_3^p(1\mathrm{P} \to 1\mathrm{S}) = |\mathbf{q}|\mu' \cdot I_3(1\mathrm{P} \to 1\mathrm{S}) \ . \tag{8.61}$$

Then $f_1(0)$ can be expressed as

$$f_1(0) = -\sqrt{\frac{3}{2}} \cdot \frac{2}{3} \cdot 4 \cdot \sqrt{2} \cdot ee_Q \cdot I_3(1\mathrm{P} \to 1\mathrm{S}) \cdot |\mathbf{q}|\mu' \ . \tag{8.62}$$

Plugging this expression of $f_1(0)$ into the formula for the decay width in Eq.(8.17), we get radiative decay width of $A(^1P_1) \to P(^1S_0) + \gamma$ in the leading order non-relativistic and no-recoil approximation:

$$\begin{aligned} \Gamma_{\mathrm{NR}} &= \frac{|\mathbf{q}|^3 \mu'^2}{12\pi M'^2} \frac{3}{2} \cdot \frac{4}{9} \cdot 16 \cdot 2 \cdot e^2 e_Q^2 |I_3(1\mathrm{P} \to 1\mathrm{S})|^2 \\ &= \left[\frac{16\mu'^2}{M'^2}\right] \cdot \frac{4}{9} \cdot \alpha e_Q^2 |\mathbf{q}|^3 \cdot |I_3(1\mathrm{P} \to 1\mathrm{S})|^2 \\ &= \frac{4}{9} \cdot \alpha e_Q^2 |\mathbf{q}|^3 \cdot |I_3(1\mathrm{P} \to 1\mathrm{S})|^2 \cdot (1 + \mathcal{O}(m_Q^{-2})) \\ &\approx \frac{4}{9} \cdot \alpha e_Q^2 |\mathbf{q}|^3 \cdot |I_3(1\mathrm{P} \to 1\mathrm{S})|^2 \ , \end{aligned} \tag{8.63}$$

where we have made use of the approximate relations of masses:

$$\mu' = \frac{m_Q}{2}\ , M' \simeq 2m_Q\ ,\ \rightarrow \left[\frac{16\mu'^2}{M'^2}\right] \simeq 1\ . \tag{8.64}$$

Eq.(8.63) matches the non-relativistic electric dipole transition formula for transition ${}^1P_1 \rightarrow {}^1S_0$ in Eq.(8.41), which proves the validity of light-front framework in the non-relativistic limit in the application to heavy quarkonium systems.

8.2.2 Next-to-leading order correction

We next include the $\mathcal{O}(m_Q^{-2})$ contributions in Eq.(8.53) with the no-recoil approximation. In this case, $f_1(q^2 \rightarrow 0)$ is given by

$$\begin{aligned}
f_1(0) &\approx -\frac{1}{4\pi}\cdot\frac{\sqrt{2}}{\beta}\cdot ee_Q \int d^3\mathbf{p}'\, R_{1\mathrm{S}}(\mathbf{p}')R_{1\mathrm{S}}(\mathbf{p}')p_\perp'^2 \\
&\times \left[2 + \frac{2M'^2}{4m_Q^2}(1-\frac{\mathbf{p}'^2}{m_Q^2}) + \frac{M''^2 - M'^2}{2m_Q^2} + \frac{2|\mathbf{q}|M'}{m_Q^2}\frac{1}{4\beta^2}p_\perp'^2\right] \\
&= -\frac{1}{4\pi}\cdot\frac{\sqrt{2}}{\beta}\cdot ee_Q \int d^3\mathbf{p}'\, R_{1\mathrm{S}}(\mathbf{p}')R_{1\mathrm{S}}(\mathbf{p}')p_\perp'^2 \\
&\times\ 4\left[1 - \frac{1}{2}\frac{\mathbf{p}'^2}{m_Q^2} + \frac{|\mathbf{q}|}{m_Q}\frac{1}{4\beta^2}p_\perp'^2 + R_{1\mathrm{P},1\mathrm{S}} + \mathcal{O}(m_Q^{-4})\right] \\
&= -\frac{1}{4\pi}\cdot\frac{\sqrt{2}}{\beta}\cdot ee_Q \int d^3\mathbf{p}'\, R_{1\mathrm{S}}(\mathbf{p}')R_{1\mathrm{S}}(\mathbf{p}') \\
&\times\ 4\left[\frac{2}{3}\cdot(1+R_{1\mathrm{P},1\mathrm{S}})\mathbf{p}'^2 - \frac{1}{3}\frac{\mathbf{p}'^4}{m_Q^2} + \frac{2}{15}\frac{|\mathbf{q}|}{m_Q}\frac{1}{\beta^2}\mathbf{p}'^4 + \mathcal{O}(m_Q^{-4})\right] \\
&= -\sqrt{3}\cdot ee_Q\cdot 4\cdot\frac{2}{3}\cdot I_3^p(1\mathrm{P}\rightarrow 1\mathrm{S}) \\
&\times \left[1 + R_{1\mathrm{P},1\mathrm{S}} - \left(\frac{1}{2}\frac{1}{m_Q^2} - \frac{1}{5}\frac{|\mathbf{q}|}{m_Q}\frac{1}{\beta^2}\right)\frac{I_5^p(1\mathrm{P}\rightarrow 1\mathrm{S})}{I_3^p(1\mathrm{P}\rightarrow 1\mathrm{S})} + \mathcal{O}(m_Q^{-4})\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{3}\cdot ee_Q\cdot 4\cdot\frac{2}{3}\cdot|\mathbf{q}|\mu'\cdot I_3(1\mathrm{P}\to 1\mathrm{S})\\
&\times \left[1+R_{1\mathrm{P},1\mathrm{S}}-|\mathbf{q}|^2\left(\frac{1}{2}\frac{\mu'^2}{m_Q^2}-\frac{1}{5}\frac{\mu'}{m_Q}\right)\frac{I_5(1\mathrm{P}\to 1\mathrm{S})}{I_3(1\mathrm{P}\to 1\mathrm{S})}+\mathcal{O}(m_Q^{-4})\right] ,
\end{aligned}
\tag{8.65}
$$

where we have made use of the symmetry property of integral for the function $F(\mathbf{p}^2)$ that has spherical symmetry:

$$
\int d^3\mathbf{p}\, F(\mathbf{p}^2)p_ip_jp_kp_l=\frac{1}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\int d^3\mathbf{p}\, F(\mathbf{p}^2)\mathbf{p}^4 , \tag{8.66}
$$

and $R_{1\mathrm{P},1\mathrm{S}}$ is given by

$$
R_{1\mathrm{P},1\mathrm{S}}=\frac{M''^2-M'^2}{8m_Q^2}+\frac{M'^2-4m_Q^2}{8m_Q^2}\sim\mathcal{O}(m_Q^{-2}) . \tag{8.67}
$$

Combining Eq.(8.65) and Eq.(8.17), we obtain the next-to-leading order ($\mathcal{O}(m_Q^{-2})$) formula for the radiative decay width for the heavy quarkonium systems (${}^1P_1\to{}^1S_0$) in the non-relativistic and no-recoil approximation:

$$
\Gamma_{\mathrm{NLO}}=\Gamma_{\mathrm{NR}}[1+R_{1\mathrm{P},1\mathrm{S}}-|\mathbf{q}|^2\left(\frac{1}{2}\frac{\mu'^2}{m_Q^2}-\frac{1}{5}\frac{\mu'}{m_Q}\right)\frac{I_5(1\mathrm{P}\to 1\mathrm{S})}{I_3(1\mathrm{P}\to 1\mathrm{S})}+\mathcal{O}(m_Q^{-4})]^2 . \tag{8.68}
$$

8.3 Analysis of radiative transitions of $h_c(1P)$ and $h_b(1P)$

In this section we apply the radiative transition formulas for the decay $1^{+-}({}^1P_1)\to 0^{-+}({}^1S_0)+\gamma$ in the framework of the light-front quark model, which we reviewed in Section 8.1, to study the radiative decay of the $c\bar{c}$ state $h_{c1}(1P)$ via the channel $h_c(1P)\to\eta_c(1S)+\gamma$ and the $b\bar{b}$ state $h_b(1P)$ via the channel $h_b(1P)\to\eta_b(1S)+\gamma$. We present the results of our numerical calculations of decay widths. Our results extend those which we previously presented with Ke and Li in [69]. For the charmonium $h_c(1P)$ radiative decay, we compare our result with experimental data on the width, as listed in the Particle Data Group Review of Particle Properties (RPP) [54].

Table 8.1: Decay width (in units of keV) of $h_c(1P) \to \eta_c(1S) + \gamma$ in the light-front quark model, denoted LFQM, as compared with experimental data from [54], denoted exp.(PDG) and predictions from other theoretical models, including non-relativistic potential model (NR) [36, 38, 52](denoted M1 [36] and M4 [52]), relativistic quark model (R) [47](M2), the Godfrey-Isgur potential model (GI) [52](M3), screened potential models with zeroth-order wavefunctions (SNR_0) and first-order relativistically corrected wavefunctions (SNR_1) [53](M5). For experimental data, we use the PDG value of the total width $\Gamma_{h_c(1P)} = 700 \pm 280$ (stat.) ± 220 (syst.) keV and $BR(h_c(1P) \to \eta_c(1S) + \gamma) = 51 \pm 6$ % [54].

Decay mode	LFQM	exp.(PDG)	M1	M2	M3	M4	M5
$h_c(1P) \to \eta_c(1S) + \gamma$	398 ± 99	357 ± 280	482	560	498	650	764
					352		323

Table 8.2: Decay width (in units of keV) of $h_b(1P) \to \eta_b(1S) + \gamma$ in the light-front quark model, denoted LFQM, as compared with predictions from other theoretical models, including non-relativistic potential model (NR) [36](denoted M1), relativistic quark model (R) [47](M2), the Godfrey-Isgur potential model (GI) [49](M3), screened potential models with zeroth-order wavefunctions (SNR_0) and first-order relativistically corrected wavefunctions (SNR_1) [48](M4) and the nonrelativistic constituent quark model (CQM) [50](M5).

Decay mode	LFQM	M1	M2	M3	M4	M5
$h_b(1P) \to \eta_b(1S) + \gamma$	37.5 ± 7.5	27.8	52.6	55.8	35.7	43.7
				36.3		

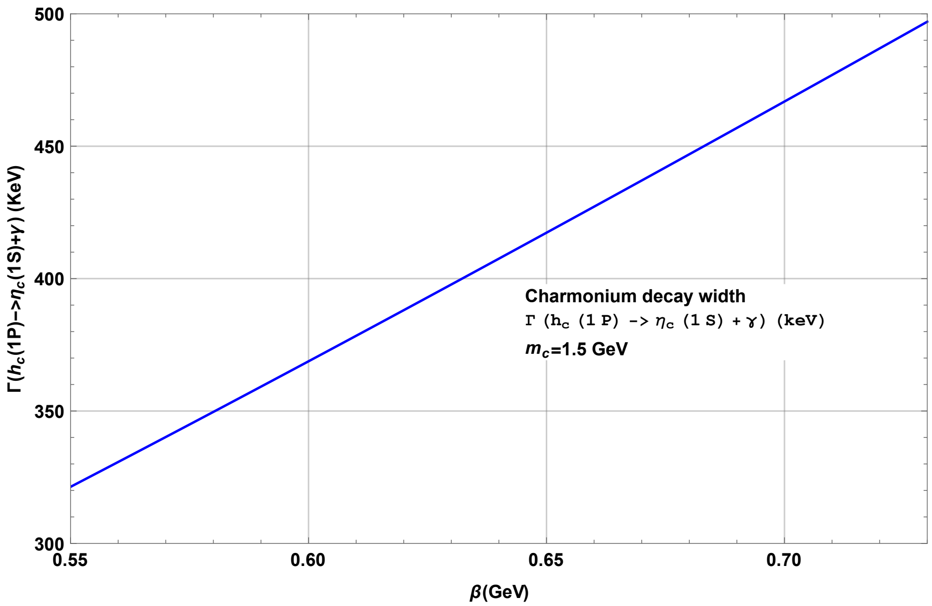

Figure 8.2: Decay width for $h_c(1P) \to \eta_c(1S) + \gamma$ (keV) as a function of $\beta_{h_c(1P)(\eta_c)(1S)}$ in LFQM, with $m_c = 1.5$ GeV.

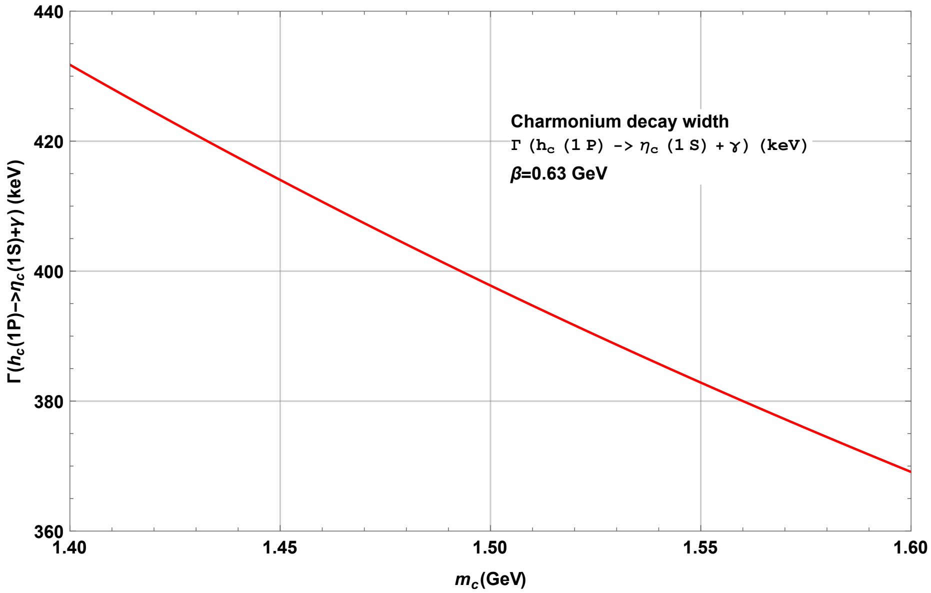

Figure 8.3: Decay width for $h_c(1P) \to \eta_c(1S) + \gamma$ (keV) as a function of m_c in the LFQM, with $\beta_{h_c(1P)(\eta_c(1S))} = 0.63$ GeV.

We also list the theoretical calculations from other models, including non-relativistic potential model (NR) [36,38,52], relativistic quark model (R) [47], the Godfrey-Isgur potential model (GI) [52], screened potential models with zeroth-order wavefunctions (SNR_0) and first-order relativistically corrected wavefunctions (SNR_1) [53].

Although the PDG lists the width for the decay $h_c(1P) \to \eta_c(1S) + \gamma$, it does not list the width for the $h_b(1P) \to \eta_b(1S) + \gamma$ decay, only the branching ratio. Since our calculation yields the width itself, and a calculation of the branching ratio requires division by the total width, we therefore compare our results on the widths for these decays with predictions from other models, including the non-relativistic potential model (NR) [36], the relativistic quark model (R) [47], the Godfrey-Isgur potential model (GI) [49], screened potential models with zeroth-order wave functions (SNR_0) and first-order relativistically corrected wave functions(SNR_1) [48], as well as the nonrelativistic constituent quark model (CQM) [50].

First, we study the radiative decay $h_c(1P) \to \eta_c(1S) + \gamma$ in the LFQM, which depends on the corresponding harmonic oscillator wavefunction ($\beta_{h_c(\eta_c)}$)

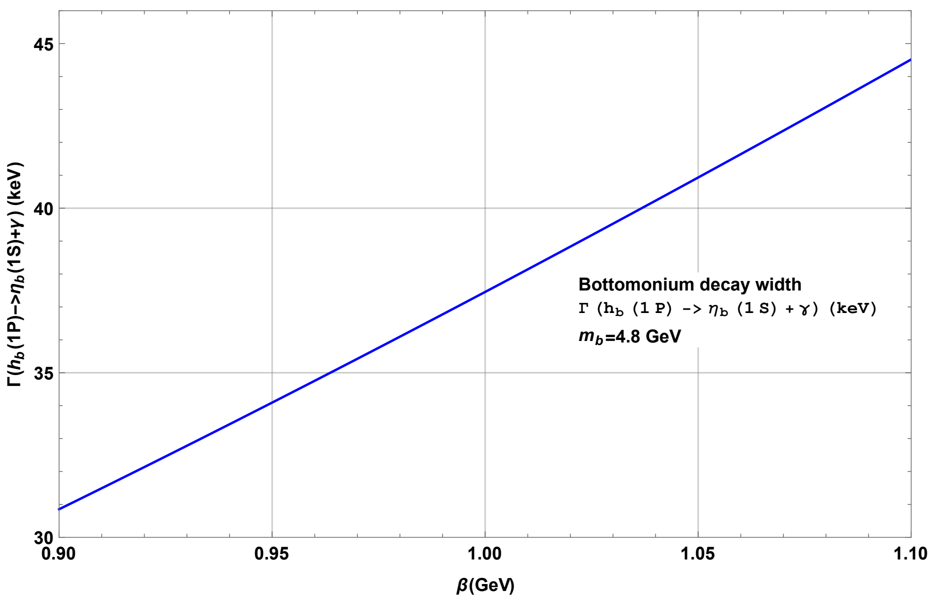

Figure 8.4: Decay width for $h_b(1P) \to \eta_b(1S) + \gamma$ (keV) as a function of $\beta_{h_b(1P)(\eta_b(1S))}$ in the LFQM, with $m_b = 4.8$ GeV.

and the effective charm quark mass, m_c. For the central values of m_c and the wavefunction parameters β, we use the central values of these parameters suggested by previous study of LFQM [70]:

$$m_c = 1.5 \pm 0.1 \text{ GeV} . \tag{8.69}$$

$$\beta_{h_c(\eta_c)} = 0.63 \pm 0.1 \text{ GeV} . \tag{8.70}$$

While Ref. [69] allowed a 10 % variation in input parameters, we investigate a somewhat larger variation, as indicated in Eqs. (8.69) and (8.70). We present our numerical results in Table 8.1, with the uncertainties arising from the uncertainties in the β parameters and the value of m_c. We also plot the predicted width as a function of the input values for the charm quark mass m_c and wavefunction structure parameter $\beta_{h_c(\eta_c)}$ in Fig. 8.2 and Fig. 8.3. From these results, we find that the main theoretical uncertainties come from variation of $\beta_{h_c(\eta_c)}$. With the same central value for $\beta_{h_c(\eta_c)}$ as was used in [69], we obtain a somewhat smaller central value for the width, namely 398 keV as contrasted with 685 keV in [69]. As is evident from our Table 8.1, our current result for this width agrees well with experimental data within

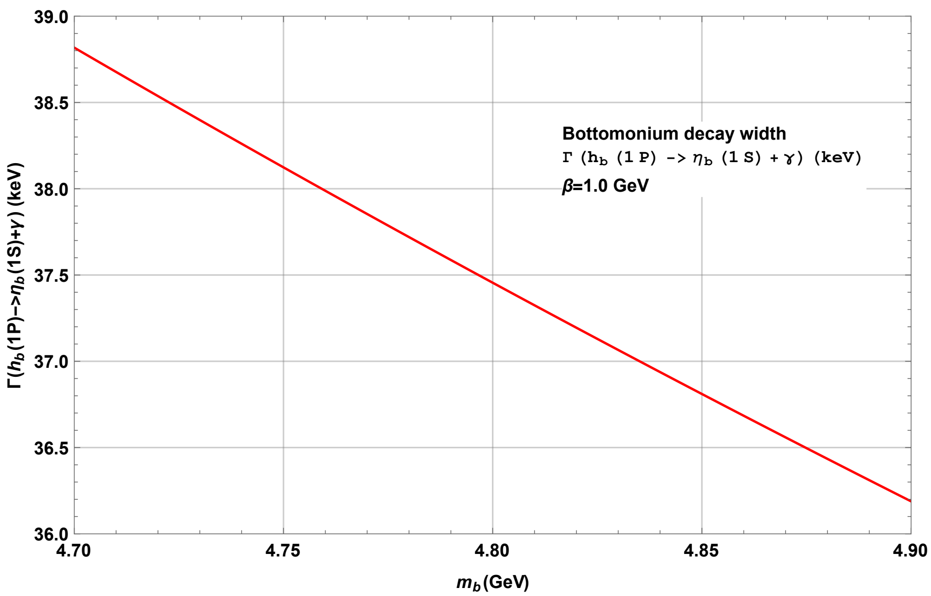

Figure 8.5: Decay width for $h_b(1P) \to \eta_b(1S) + \gamma$ (keV) as a function of m_b in the LFQM, with $\beta_{h_b(1P)(\eta_b(1S))} = 1.0$ GeV.

the range of experimental and theoretical uncertainties. The experimental data have substantial uncertainties, and our result is relatively close to the central experimental value, compared to other non-relativistic models. The reason that our current calculation of the width $\Gamma(h_c(1P) \to \eta_c(1S) + \gamma)$ yields a smaller result than that obtained in Ref. [43] may be due to the fact that the numerical integration that is necessary in the calculation of the amplitude involves significant cancellations between different terms, and our current numerical integration routine uses higher precision than was used in parts of the previous calculation in Ref. [43].

Next we study radiative decay of $h_b(1P) \to \eta_b(1S) + \gamma$ in LFQM. For the central value of the effective bottom/beauty quark mass m_b, we use the value suggested by the previous LFQM study [69] (see also [70]). For the wavefunction parameter $\beta_{h_b(1P)(\eta_b(1S))}$, we estimate this to be in the range $\beta \sim 0.9 - 1.3$ GeV, which is suggested in [49], where β is fitted by equating the rms radius of the harmonic oscillator wavefunction for the specified states with the rms radius of the wavefunctions calculated using the relativized quark model. Our values for these input parameters are:

$$m_b = 4.8 \pm 0.1 \text{ GeV} . \tag{8.71}$$

$$\beta_{h_b(1P)(\eta_b(1S))} = 1.0 \pm 0.1 \text{ GeV} . \tag{8.72}$$

We list the numerical results in the LFQM in Table 8.2. For comparison, we also list other theoretical calculations from various types of models, including the non-relativistic potential model (NR) [36], the relativistic quark model (R) [47], the Godfrey-Isgur potential model (GI) [49], screened potential models with zeroth-order wavefunctions (SNR_0) and first-order relativistically corrected wavefunctions(SNR_1) [48] and the non-relativistic constituent quark model (CQM) [50]. As can be seen from Table 8.2, with the given range of uncertainties, our value agrees with predictions from the non-relativistic potential model (NR) [36], the Godfrey-Isgur potential model (GI) [49] and screened potential models with relativistically corrected wavefunctions (SNR_1) [48]. To show the theoretical uncertainties arising from uncertainties in the $\beta_{h_b(1P)(\eta_b(1S))}$ parameter and the value of m_b, we also plot the decay width for $h_b(1P) \to \eta_b(1S) + \gamma$ as a function of these parameters in Fig. 8.4 and Fig. 8.5. We find that the width is not very sensitive to the variation of m_b and the main uncertainties arise from the uncertainty in the wavefunction parameter $\beta_{h_b(1P)(\eta_b(1S))}$.

These results show that the light-front quark model with phenomenological meson wavefunctions (specifically, harmonic oscillator wavefunctions) is suitable for the calculation of quarkonium ${}^1P_1 \to {}^1S_0 + \gamma$ radiative decay widths, since this model gives reasonable predictions for these widths, as compared with experimental data and other theoretical models.

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[100] Here and elsewhere, when an expression is given for N and/or p that formally evaluates to a non-integral real value, it is understood implicitly that one infers an appropriate integral value from it, either the greatest integer smaller than the real value, or the least integer greater than the real value, or the closest integer, depending on the context.

[101] Note that for all of the three new chiral gauge theories that we construct and study, namely the Adj, AT, and $S\bar{S}$ theories, when we take a large-N limit, we keep p fixed rather than having it increase like rN. The reason for this difference relative to the Sp theory is that since each of these three new theories involves p fermions in two-index representations of SU(N), it follows that for these fermions, $T(R) \propto N$ for large N, in contrast to the Sp model, where the p fermions are in the

F or $\bar{F}$ representation, with $T(F) = 1/2$ independent of N. Hence, to keep b_1 positive as required for asymptotic freedom, the large-N limit for these three theories is taken with p fixed.

[102] We recall that an analogous vacuum alignment argument is used in quantum chromodynamics to infer that among the various $3 \times \bar{3} \to 1$ quark condensates, $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$, and $\langle \bar{s}s \rangle$ form, but others such as $\langle \bar{u}d \rangle$ and $\langle \bar{u}s \rangle$, and $\langle \bar{d}s \rangle$ do not.

[103] Parenthetically, we note that in the *Adj* theory, the anomaly in the global $\mathrm{U}(1)_{R_{sc}}$ symmetry is analogous to the anomaly in the $\mathrm{U}(1)_R$ global symmetry in supersymmetric $\mathrm{SU}(N)$ gauge theory.

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Appendix A

Beta Function Coefficients and Relevant Group Invariants

For reference, we list the one-loop and two-loop coefficients [75,76] in the beta function (3.1) for a non-Abelian chiral gauge theory with gauge group G and a set of chiral fermions comprised of N_i fermions transforming according to the representations $\{R_i\}$.

$$b_1 = \frac{1}{3}\Big[11C_2(G) - 2\sum_{R_i} N_i T(R_i)\Big] \tag{A.1}$$

and

$$b_2 = \frac{1}{3}\Big[34C_2(G)^2 - 2\sum_{R_i} N_i\{5C_2(G) + 3C_2(R_i)\}T(R_i)\Big] \ . \tag{A.2}$$

For a representation R, the Casimir invariants $C_2(R)$ and $T(R)$ are defined as

$$\sum_{i,j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij}\mathcal{D}_R(T_b)_{ji} = T(R)\delta_{ab} \tag{A.3}$$

and

$$\sum_{a=1}^{o(G)} \sum_{j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij}\mathcal{D}_R(T_a)_{jk} = C_2(R)\delta_{ik} \ , \tag{A.4}$$

where T_a are the generators of G, and $\mathcal{D}_R$ is the matrix representation of R. These satisfy

$$T(R)\, o(G) = C_2(R)\dim(R) \ , \tag{A.5}$$

where $o(G) = N^2 - 1$ for SU(N) and dim(R) is the dimension of the representation R.

We list below the group invariants that we use for the relevant case $G =$ SU(N). We have $C_2(G) = C_2(Adj) = T(Adj) = N$, and, as in the text, we use the symbols F for $\square$ and S for $\square\square$. The symmetric and antisymmetric rank-k representations of SU(N) are denoted S_k and $A_k \equiv [k]_N$. In terms of Young tableaux, $S_1 = A_1 = \square$, $S_2 = \square\square$, $A_2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, etc. (In the text, where no confusion would result, we denote $S_2 \equiv S$ and $A_2 \equiv A$.) We have

$$C_2(F) = \frac{N^2 - 1}{2N} \ , \quad T(F) = \frac{1}{2} \ , \tag{A.6}$$

$$C_2(S) = \frac{(N+2)(N-1)}{N} \ , \quad T(S) = \frac{N+2}{2} \ , \tag{A.7}$$

$$C_2([k]_N) = \frac{k(N+1)(N-k)}{2N} \ , \tag{A.8}$$

and

$$T([k]_N) = \frac{1}{2}\binom{N-2}{k-1} \ . \tag{A.9}$$

Hence, for our case $N = 2k$,

$$C_2([k]_{2k}) = \frac{k(2k+1)}{4} \tag{A.10}$$

and

$$T([k]_{2k}) = \frac{(2k-2)!}{2[(k-1)!]^2} \ . \tag{A.11}$$

For the adjoint representation, $C_2(adj) \equiv C_2(G) = T(Adj) = N$. For the rank-$k$ symmetric and antisymmetric representations S_k and A_k,

$$T(S_k) = \frac{\prod_{j=2}^{k}(N+j)}{2(k-1)!} \tag{A.12}$$

$$T(A_k) = \frac{1}{2}\binom{N-2}{k-1} = \frac{\prod_{j=2}^{k}(N-j)}{2(k-1)!} \tag{A.13}$$

$$C_2(S_k) = \frac{k(N+k)(N-1)}{2N} \tag{A.14}$$

and

$$C_2(A_k) = \frac{k(N-k)(N+1)}{2N} \ . \tag{A.15}$$

Hence, in particular, with T_2 standing for the rank-2 tensor representation S_2 (+ sign) or A_2 (− sign) here and below, one has

$$T(T_2) = \frac{N \pm 2}{2} \tag{A.16}$$

$$C_2(T_2) = \frac{(N \pm 2)(N \mp 1)}{N} \tag{A.17}$$

$$T(T_3) = \frac{(N \pm 2)(N \pm 3)}{4} \tag{A.18}$$

and

$$C(T_3) = \frac{3(N \pm 3)(N \mp 1)}{2N} \ . \tag{A.19}$$

The anomaly produced by chiral fermions transforming according to the representation R of a group G is defined as

$$\mathrm{Tr}_R(T_a, \{T_b, T_c\}) = \mathcal{A}(R) d_{abc} \tag{A.20}$$

where the d_{abc} are the totally symmetric structure constants of the corresponding Lie algebra. Groups for which $= \mathcal{A}(R) = 0$ include those with real or pseudoreal representations, SO$(4k+2)$ for $k \geq 2$, and E_6 [104, 106, 114]. For convenience, we may define $\mathcal{A}(\Box) = 1$ for SU(N). For S_k and A_k [106],

$$\mathcal{A}(S_k) = \frac{(N+k)!(N+2k)}{(N+2)!(k-1)!} \tag{A.21}$$

and, for $1 \leq k \leq N-1$,

$$\mathcal{A}(A_k) = \frac{(N-3)!(N-2k)}{(N-k-1)!(k-1)!} \ . \tag{A.22}$$

(Note that $[N]_N$ is the singlet, so $\mathcal{A}([N]_N) = 0$.) Hence, in particular,

$$\mathcal{A}(T_2) = N \pm 4 \tag{A.23}$$

$$\mathcal{A}(T_3) = \frac{(N \pm 3)(N \pm 6)}{2} \ . \tag{A.24}$$

$$\mathcal{A}([2]_N) = N - 4 \ , \tag{A.25}$$

$$\mathcal{A}([3]_N) = \frac{(N-3)(N-6)}{2} \ , \tag{A.26}$$

$$\mathcal{A}([4]_N) = \frac{(N-3)(N-4)(N-8)}{3!} \ . \tag{A.27}$$

From Eq. (A.22), there follows the recursion relation

$$\mathcal{A}([k]_N) + \mathcal{A}([k+1]_N) = \mathcal{A}([k+1]_{N+1}) \quad \text{for } 1 \le k \le N-1 \ . \tag{A.28}$$

The rank of SO(N) is the integral part of $N/2$. We denote A_t the rank-t antisymmetric tensor representation, with dimension $\binom{N}{t}$, where $\binom{a}{b} = a!/[b!(a-b)!]$. Note that for SO($N$), the adjoint representation is the same as A_2 and the vector, fundamental, and A_1 representations are the same. With an appropriate normalization convention for the generators of SO(N) (which does not affect the physics), one has [104, 115]

$$T(adj) = C_2(adj) = N - 2 \ , \tag{A.29}$$

$$T(\Box) = 1 \ , \tag{A.30}$$

and

$$C_2(\Box) = \frac{N-1}{2} \ . \tag{A.31}$$

For SO(N) with $N = 2r$ and $\mathcal{S}$ the spinor representation,

$$\dim(\mathcal{S}) = 2^{r-1} \ , \tag{A.32}$$

$$T(\mathcal{S}) = 2^{r-4} \ , \tag{A.33}$$

$$C_2(\mathcal{S}) = \frac{r(2r-1)}{8} \ . \tag{A.34}$$

Denoting the antisymmetric rank-t tensor representation of SO($2r$) as A_t, one has

$$C_2(A_t) = \frac{t(2r-t)}{2} \ . \tag{A.35}$$

A gauge theory in $d = 4$ dimensions with gauge group G contains instantons if $\pi_{d-1}(G) = \pi_3(G)$ is nontrivial. One has [117]

$$\pi_3(\mathrm{SU}(N)) = \mathbb{Z} \tag{A.36}$$

and

$$\pi_3(\mathrm{SO}(N)) = \mathbb{Z} \quad \text{if } N \geq 5 \ . \tag{A.37}$$

The global anomaly in an $\mathrm{SU}(2)_L$ gauge theory is due to

$$\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2 \ , \tag{A.38}$$

Further,

$$\pi_4(\mathrm{SU}(N)) = \emptyset \quad \text{if } N \geq 3 \tag{A.39}$$

and

$$\pi_4(\mathrm{SO}(N)) = \emptyset \quad \text{if } N \geq 6 \ . \tag{A.40}$$

Appendix B

The wavefunctions

The normalization of the S-wave meson wavefunction in the light-front framework is

$$\frac{1}{2(2\pi)^3}\int dx_2 dp_\perp^2\, |\varphi(x_2,p_\perp)|^2 = 1. \tag{B.1}$$

Here $\varphi(x_2,p_\perp)$ is related to the wavefunction in normal coordinates $\psi(p)$ by

$$\varphi(x_2,p_\perp) = 4\pi^{\frac{3}{2}}\sqrt{\frac{dp_z}{dx_2}}\psi(p)\ , \quad \frac{dp_z}{dx_2} = \frac{e_1' e_2}{x_1 x_2 M_0'}\ . \tag{B.2}$$

The normalization of $\psi(p)$ is given by

$$\int d\mathbf{p}^3\, |\psi(p)|^2 = 4\pi\int p^2 dp\, |\psi(p)|^2 = 1\ . \tag{B.3}$$

The normalization for the P-wave meson wavefunction in the light-front framework is [65]

$$\frac{1}{2(2\pi)^3}\int dx_2 dp_\perp^2\, |\varphi_p(x_2,p_\perp)|^2 p_i p_j = \delta_{ij}\ , \tag{B.4}$$

where $p_i = (p_x, p_y, p_z)$. In terms of the P-wave wavefunction in normal coordinates,

$$\varphi_p(x_2,p_\perp) = 4\pi^{\frac{3}{2}}\sqrt{\frac{dp_z}{dx_2}}\psi_p(p)\ , \quad \frac{dp_z}{dx_2} = \frac{e_1' e_2}{x_1 x_2 M_0'}\ . \tag{B.5}$$

we have the following normalization condition:

$$\frac{1}{3}\cdot 4\pi\int_0^\infty |\psi_p(p)|^2 p^4 dp = 1\ . \tag{B.6}$$

For the gaussian type 1P and 1S wavefunctions, we have the relation

$$\psi_p(p) = \sqrt{\frac{2}{\beta^2}}\psi(p) \ . \tag{B.7}$$

The explicit form of 1-S harmonic oscillator wavefunction in the light-front approach is given by [65]

$$\psi(p) \ = \ \left(\frac{1}{\beta^2\pi}\right)^{\frac{3}{4}} \exp\left(-\frac{1}{2}\frac{p^2}{\beta^2}\right) \ . \tag{B.8}$$

Appendix C

Some expressions in the light-front formalism

In the covariant light-front formalism we have

$$\begin{aligned}
M_0'^2 &= (e_1' + e_2)^2 = \frac{p_\perp'^2 + m_1'^2}{x_1} + \frac{p_\perp'^2 + m_2^2}{x_2} \\
M_0''^2 &= (e_1'' + e_2)^2 = \frac{p_\perp''^2 + m_1''^2}{x_1} + \frac{p_\perp''^2 + m_2^2}{x_2} \\
\tilde{M}_0' &= \sqrt{M_0'^2 - (m_1' - m_2)^2} \\
\tilde{M}_0'' &= \sqrt{M_0''^2 - (m_1'' - m_2)^2} \\
p_z' &= \frac{x_2 M_0'}{2} - \frac{m_2^2 + p_\perp'^2}{2 x_2 M_0'} \\
p_z'' &= \frac{x_2 M_0''}{2} - \frac{m_2^2 + p_\perp''^2}{2 x_2 M_0''}.
\end{aligned} \tag{C.1}$$

The explicit expressions for $A_j^{(i)}(i,j=1\sim 4)$ and Z_2 are

$$\begin{aligned}
&A_1^{(1)}=\frac{x_1}{2}\ ,\ A_2^{(1)}=A_1^{(1)}-\frac{p'_\perp\cdot q_\perp}{q^2}\ ,\\
&A_1^{(2)}=-p'^2_\perp-\frac{(p'_\perp\cdot q_\perp)^2}{q^2},\ A_2^{(2)}=(A_1^{(1)})^2,\\
&A_3^{(2)}=A_1^{(1)}A_2^{(1)}\ ,\ A_4^{(2)}=(A_2^{(1)})^2-\frac{1}{q^2}A_1^{(2)}\ ,\\
&A_1^{(3)}=A_1^{(1)}A_1^{(2)}\ ,A_2^{(3)}=A_2^{(1)}A_1^{(2)}\ ,\\
&A_3^{(3)}=A_1^{(1)}A_2^{(2)}\ ,A_4^{(3)}=A_2^{(1)}A_2^{(2)}\ .
\end{aligned}\tag{C.2}$$

$$Z_2=\hat{N}'_1+m'^2_1-m_2^2+(1-2x_1)M'^2+(q^2+q\cdot P)\frac{p'_\perp\cdot q_\perp}{q^2}.\tag{C.3}$$